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Abstract

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GEOPHYSICS

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ON SOME SIMPLIFICATIONS OF THE EQUATIONS OF DYNAMICS OF STATIONARY CURRENTS IN A BAROCLINIC SEA

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We consider the problem of established currents, generated by wind, in a deep, stably stratified sea with vertical shores. Let the y-axis be directed northward, the x-axis eastward, and the z-axis vertically downward. Suppose that, in the absence of currents, the vertical distribution of density and pressure is given by the formulas:

$$\mathfrak{P}(z) = \rho_0(1+\gamma z), \qquad P(z) = p_a + \int_0^z g\mathfrak{P}(z)\,dz,$$

where $\rho_0,\ \gamma,\ p_a$ are constants.

For the description of this problem in the dynamics of sea currents the following equations are adopted:

$$-fv = -\frac{\partial p}{\partial x} + A_z\frac{\partial^2 u}{\partial z^2} + A_l\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right); \tag{1}$$

$$fu = -\frac{\partial p}{\partial y} + A_z\frac{\partial^2 v}{\partial z^2} + A_l\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right); \tag{2}$$

$$g\rho = \frac{\partial p}{\partial z}; \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0; \tag{4}$$

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\left(\frac{\partial \rho}{\partial z} + \gamma\right) = A_z\frac{\partial^2 \rho}{\partial z^2} + A_l\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right). \tag{5}$$

Here $u,\ v,\ w$ are the velocity components along the axes $x,\ y,\ z$, respectively; $\rho = \rho'/\rho_0$; $p = p'/\rho_0$, where ρ' and p' are the deviations of density and pressure from $\mathfrak{P}(z)$ and $P(z)$, the values of these elements in the sea at rest; f is the Coriolis parameter; g is the acceleration of gravity; A_z and A_l are constant coefficients of vertical and horizontal turbulent exchange.

The boundary conditions of the problem are the following:

for $z = 0$:

$$A_z \frac{\partial u}{\partial z}(x, y, 0) = -T_x, \qquad A_z \frac{\partial v}{\partial z}(x, y, 0) = -T_y, \qquad \frac{\partial \rho}{\partial z} = 0; \tag{6}$$

$$w(x, y, 0) = 0; \tag{7}$$

$$p(x, y, 0) = -g\zeta(x, y); \tag{8}$$

for $z = \infty$:

$$u = v = w = p = \rho = 0. \tag{9}$$

Here

$$T_x = \frac{1}{\rho_0} T'_x; \qquad T_y = \frac{1}{\rho_0} T'_y,$$

where T'_x, T'_y are the prescribed tangential wind stresses; $\zeta(x, y)$ is the sea level, determined from equation (8).

On the vertical walls $u = 0$, $v = 0$, $\partial\rho/\partial\nu = 0$, where ν is the direction of the normal to the wall.

The aim of the present work is to simplify equations (1)–(5) and the boundary conditions of the problem by estimating the order of magnitude of the quantities entering expressions (1)–(9), for motions with characteristic horizontal scale $L = 1000$ km. Such a value of L is adopted in the consideration of oceanic currents.

Let us write down the determining parameters of the problem (all orders given below are indicated in the CGS system): $f = 10^{-4}$; $\beta = 10^{-13}$; $T = 1$; $\gamma = 10^{-7}$; $g = 10^3$; $A_z = 10^2$; $A_l = 10^9$; $L = 10^8$; β is the latitudinal variation of the Coriolis parameter; T is the characteristic value of the wind stress (T_x, T_y). The value A_l is computed by means of the Richardson–Obukhov formula: $A_l = kL^{4/3}$, where $k \sim 0.02$. We shall denote the characteristic values for the horizontal and vertical velocity, density, and pressure respectively by U, W, σ, Π.

We shall decompose the field of hydrodynamic elements in the sea into the sum of two components: the drift component $(u_\mathrm{d}, v_\mathrm{d}, w_\mathrm{d}, p_\mathrm{d}, \rho_\mathrm{d})$ and the gradient

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component $(u_r, v_r, w_r, p_r, \rho_r)$, in whose formation different physical factors participate. The component $(u_d, v_d, w_d, p_d, \rho_d)$ is the result of the direct action of wind stress on the water surface. For it, vertical mixing plays an essential role. The penetration depth of the drift component h is of the order of 10 m—the order of the "friction depth."

The gradient component $(u_r, v_r, w_r, p_r, \rho_r)$ is a secondary result of the action of the wind and is essentially connected with density inhomogeneity. The penetration depth of this component H is of the order of 1000 m—the order of the thickness of the baroclinic layer in the sea.

We shall determine the orders of magnitude separately for the drift and gradient components.

Let us begin with the drift component. Since $\partial u_d/\partial z \gg \partial u_r/\partial z$; $\partial v_d/\partial z \gg \partial v_r/\partial z$, then $U_d = Th/A_z$. In equations (1) and (2), the Coriolis force and vertical friction must have the same order. Assuming that in equation (5) γw_d and $A_z \partial^2 \rho_d/\partial z^2$ are of the same order, and in equation (4) all terms are of the same order, we have

$$U_d = \frac{Th}{A_z} = 10, \qquad W_d = U_d \frac{h}{L} = 10^{-4}, \qquad h = \sqrt{\frac{A_z}{f}} = 10^3,$$

$$\sigma_d = \frac{\gamma W_d h^2}{A_z} = 10^{-7}, \qquad \Pi_d = gh\sigma_d = 10^{-1}, \qquad \frac{\Pi_d}{L} = 10^{-9}.$$

We proceed to determine the scales for the gradient component. In this case, in equations (1) and (2), the Coriolis force and the pressure gradient must have the same order. In equation (5), horizontal advection has the same order as the term γw_r. By virtue of condition (7), $W_r = W_d$. We have

$$U_r = \sqrt{\frac{\gamma g H T}{L f^2}} \simeq 3, \qquad \sigma_r = \sqrt{\frac{T\gamma L}{gH}} \simeq 3 \cdot 10^{-4}, \qquad \frac{\Pi_r}{L} = \sqrt{\frac{Tg\gamma H}{L}} \simeq 3 \cdot 10^{-4}.$$

As the estimate shows, the principal role in the expression $\partial u_r/\partial x + \partial v_r/\partial y$ is played by the latitudinal variation of the Coriolis parameter β. Requiring that $\partial u_r/\partial x + \partial v_r/\partial y$ have the same order as $\partial w_r/\partial z$, we find H:

$$H = \left[\frac{f}{\beta}\sqrt{\frac{T}{Lg\gamma}}\right]^{2/3}. \tag{10}$$

Formula (10) gives the order of the thickness of the baroclinic layer in the sea, 10^5 cm. Let us note that if, in equation (5), the principal terms are taken to be lateral mixing and γw_r, or vertical mixing and γw_r, then one can

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in the same way one can obtain other expressions for H, which were obtained in the works of Lineikin ([1]) and Stommel and Veronis ([2]). The orders of H obtained from all these formulas turn out to be the same (for the adopted values of the determining parameters).

The estimates given show that $U_\text{g} \approx U_\text{d}$, $W_\text{g} \approx W_\text{d}$, $\sigma_\text{g} \gg \sigma_\text{d}$, $\Pi_\text{g} \gg \Pi_\text{d}$. Therefore we have

$$u = u_\text{g} + u_\text{d}, \qquad v = v_\text{g} + v_\text{d}, \qquad w = w_\text{g} + w_\text{d}, \qquad p = p_\text{g}, \qquad \rho = \rho_\text{g}. \quad (11)$$

If we exclude from consideration the narrow coastal strip, then, after carrying out all the estimates, we arrive at the following relations:

$$-fv_\text{d} = A_z \frac{\partial^2 u_\text{d}}{\partial z^2}, \qquad fu_\text{d} = A_z \frac{\partial^2 v_\text{d}}{\partial z^2}; \quad (12)$$

$$\frac{\partial u_\text{d}}{\partial x} + \frac{\partial v_\text{d}}{\partial y} + \frac{\partial w_\text{d}}{\partial z} = 0; \quad (13)$$

$$\text{for } z = 0 \qquad A_z \frac{\partial u_\text{d}}{\partial z} = -T_x, \qquad A_z \frac{\partial v_\text{d}}{\partial z} = -T_y; \quad (14)$$

$$\text{for } z = \infty \qquad u_\text{d} = v_\text{d} = w_\text{d} = 0. \quad (15)$$

Hence we easily find

$$w_\text{d}(x, y, 0) = \frac{\partial}{\partial x}\left(\frac{T_y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{T_x}{f}\right).$$

To determine the gradient component (the problem of the deep circulation) we have:

$$-fv_\text{g} = -\frac{\partial p}{\partial x}, \qquad fu_\text{g} = -\frac{\partial p}{\partial y}; \quad (16)$$

$$\frac{\partial p}{\partial z} = g\rho; \quad (17)$$

$$\frac{\partial u_\text{g}}{\partial x} + \frac{\partial v_\text{g}}{\partial y} + \frac{\partial w_\text{g}}{\partial z} = 0; \quad (18)$$

$$u_\text{g} \frac{\partial \rho}{\partial x} + v_\text{g} \frac{\partial \rho}{\partial y} + \gamma w_\text{g} = A_l \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right). \quad (19)$$

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The boundary conditions of the problem are:

$$\text{for } z=0 \qquad w_{\mathrm{g}} = -w_{\mathrm{d}} = \frac{\partial}{\partial y}\left(\frac{T_x}{f}\right) - \frac{\partial}{\partial x}\left(\frac{T_y}{f}\right);$$

$$\text{for } z=\infty \qquad u_{\mathrm{g}} = v_{\mathrm{g}} = w_{\mathrm{g}} = p = \rho = 0;$$

on the vertical wall there is no normal derivative of the density; the density ρ is regarded as known on the given vertical.

After solving equations (12)–(19), the sought quantities u, v, w, p, ρ are found from formulas (11).

Let us dwell on the analysis of the deep circulation. In this case it turns out that all the sought functions $u_{\mathrm{g}}, v_{\mathrm{g}}, w_{\mathrm{g}}, p, \rho$ can be expressed in terms of a single function $\psi(x, y, z)$ by the formulas

$$u_{\mathrm{g}} = -\frac{1}{f}\frac{\partial^2\psi}{\partial y\,\partial z}, \qquad v_{\mathrm{g}} = \frac{1}{f}\frac{\partial^2\psi}{\partial x\,\partial z}, \qquad w_{\mathrm{g}} = \frac{\beta}{f^2}\frac{\partial\psi}{\partial x},$$

$$p = \frac{\partial\psi}{\partial z}, \qquad \rho = \frac{1}{g}\frac{\partial^2\psi}{\partial z^2}. \tag{20}$$

To determine ψ, one obtains the equation

$$\frac{1}{f}\left[\frac{\partial^2\psi}{\partial x\,\partial z}\frac{\partial^3\psi}{\partial z^2\partial y} - \frac{\partial^2\psi}{\partial y\,\partial z}\frac{\partial^3\psi}{\partial z^2\partial x}\right] + \frac{g\gamma\beta}{f^2}\frac{\partial\psi}{\partial x} = A_l\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2\psi}{\partial z^2} \tag{21}$$

with the boundary conditions:

$$\text{for } z=0 \qquad \frac{\beta}{f^2}\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial y}\left(\frac{T_x}{f}\right) - \frac{\partial}{\partial x}\left(\frac{T_y}{f}\right);$$

$\psi = 0$ for $z = \infty$, and $\partial\psi/\partial\nu = 0$ on vertical walls; ψ is prescribed on one vertical.

Solving equation (21) presents great mathematical difficulties. For this reason, in order to obtain at least a qualitative idea of the solution, we shall discard (as is usually done in works on ocean currents) the nonlinear terms and consider the equation

$$\frac{g\gamma\beta}{f^2}\frac{\partial\psi}{\partial x} = A_l\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2\psi}{\partial z^2} \tag{22}$$

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with the same boundary conditions as for equation (21).

The solution of this equation is readily found in the following particular case. Consider a channel, elongated in the zonal direction, of width L. Let the wind be specified in the form $T_y = 0$, $T_x = T_0 \sin \lambda x \left(1 + \cos \frac{2\pi y}{L}\right)$. Then the solution of equation (22) is given by the formula:

$$\psi = \frac{2\pi T_0 f}{\lambda \beta L} e^{-qz} \cos(\lambda x + qz) \sin \frac{2\pi y}{L}, \tag{23}$$

where

$$q = \left\{ \frac{1}{2} \frac{g\gamma\beta\lambda}{A_l f^2 \left(\lambda^2 + 4\pi^2/L^2\right)} \right\}^{1/3}.$$

From formula (23) it is evident that for each z the wave in the direction of the x-axis is shifted in phase to the west, and the shift increases with depth.

In paper [3] an exact solution of equations (1)–(5) was given for a channel with $L = 100$ km and without taking nonlinear terms into account. This solution is obtained with a sufficient degree of accuracy from equations (12)–(19), if lateral friction is taken into account in equations (16) (since P. S. Lineikin considered in [3] the case $\beta = 0$). Naturally, the term $u_r \partial\rho/\partial x + v_r \partial\rho/\partial y$ must be omitted from (19), since it was not taken into account in paper [3].

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CITED LITERATURE

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[2] H. Stommel, G. Veronis, Tellus, **9**, 3 (1957).
[3] P. S. Lineikin, *Basic Problems of the Dynamic Theory of the Baroclinic Layer of the Sea*, 1957.

Note: Figure translations are in progress. See original paper for figures.

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