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Abstract

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ON THE QUESTION OF AXISYMMETRIC MOTIONS OF A FLUID

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Up to the present time, exact analytic solutions of the equations of axisymmetric flow past bodies symmetric with respect to the axis of a pipeline have not been obtained; solutions have been obtained only for certain special cases. In the present work two methods are proposed for solving the above problem: the method of Bessel polynomials and the method of finite differences.

The method of Bessel polynomials. Since the motion under consideration is symmetric with respect to the axis Oz, the velocity potential $\varphi(r, z)$ and the stream function $\psi(r, z)$ (r is the coordinate measured along the radius of the pipe; z is the longitudinal coordinate) must satisfy the differential equations

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = 0,$$

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = 0.$$

The solution of these equations under the corresponding boundary conditions has the form (R is the radius of the pipeline)

$$\varphi = \sum_{i=1}^{n} c_i J_0\left(\beta_i \frac{r}{R}\right) e^{-\beta_i |z|/R},$$

$$\psi = \sum_{i=1}^{n} c_i J_1\left(\beta_i \frac{r}{R}\right) e^{-\beta_i |z|/R},$$

where J_0 and J_1 are Bessel functions of the first kind of orders zero and one; β_i are the roots of the function J_1, i.e., the solutions of the equation

$$J_0'(\beta_i) = -J_1(\beta_i) = 0.$$

If a rectilinear-parallel flow with velocity V_∞ is imposed on the motion, then

$$\varphi = -V_\infty z + \sum_{i=1}^{n} c_i J_0\left(\beta_i \frac{r}{R}\right) e^{-\beta_i |z|/R}, \tag{1}$$

$$\psi = \frac{V_\infty r^2}{2} + r \sum_{i=1}^{n} c_i J_1\left(\beta_i \frac{r}{R}\right) e^{-\beta_i |z|/R}. \tag{2}$$

We transform these equations to dimensionless variables according to the relations

$$\varphi^* = \frac{\varphi}{V_\infty R}, \qquad \psi^* = \frac{\psi}{V_\infty R}, \qquad k_i = \frac{c_i}{V_\infty R}, \qquad u^* = \frac{u}{V_\infty}, \qquad v^* = \frac{v}{V_\infty},$$

$$\xi = \frac{r}{R}, \qquad \eta = \frac{z}{R}.$$

We have

$$\varphi^* = -\eta + \sum_{i=1}^{n} k_i J_0(\beta_i \xi) e^{-\beta_i |\eta|}, \tag{3}$$

$$\psi^* = +\frac{\xi^2}{2} + \xi \sum_{i=1}^{n} k_i J_1(\beta_i \xi) e^{-\beta_i |\eta|}. \tag{4}$$

The velocity components u^* and v^* are obtained from equations (3) and (4) in the form

$$u^* = \frac{\partial \varphi^*}{\partial \xi} = \frac{1}{\xi}\frac{\partial \psi^*}{\partial \eta} = -\sum_{i=1}^{n} k_i \beta_i J_1(\beta_i \xi) e^{-\beta_i |\eta|},$$

$$v^* = \frac{\partial \varphi^*}{\partial \eta} = -\frac{1}{\xi}\frac{\partial \psi^*}{\partial \xi} = -1 - \sum_{i=1}^{n} k_i \beta_i J_0(\beta_i \xi) e^{-\beta_i |\eta|}.$$

If a body symmetric with respect to the axis of the pipeline is placed in the pipeline, then on the surface of this body the condition $\psi^*(\xi, \eta) = 0$ is satisfied. If this condition is written for the points (ξ_j, η_j), then, using equation (4), we obtain a system of linear equations with n unknowns k_i

$$\sum_{i=1}^{n} k_i J_1(\beta_i \xi_j) e^{-\beta_i |\eta_j|} = -\frac{\xi_j}{2} \quad (j = 1, 2, \dots, n). \tag{5}$$

Fig. 1.

Figure 1: Fig. 1.

An analogous method, applied to harmonic zonal spherical functions, was used by Smith ([1]) for studying potential flows past bodies of revolution in an infinite space.

Fig. 1.

The solution of the system of equations (5) makes it possible approximately to express the result in the form of a polynomial whose terms are Bessel functions of the first kind of orders zero and one. Such a solution exactly satisfies the conditions at infinity on the walls of the pipeline and approximately the boundary conditions on the contour of the axisymmetric body. The passage to the boundary conditions for obtaining the exact solution, when choosing an increasing number of points where the stream function is required to vanish, is the subject of a separate investigation.

As an example, streamlines were found for the case of outflow through circular diaphragms of radius equal to 1/2 (Fig. 1). Since in this case $\psi^* = 1/2$ for $\xi = 1$ and $\eta = 0$, $1/2 \leq \xi \leq 1$, the coefficients k_i will be determined by solving a system of n linear equations with n unknowns k_i, written for n points lying on the diaphragm (in the example $n = 8$),

$$\sum_{i=1}^{n} k_i J_1(\beta_i \xi_j) = \frac{1-\xi_j^2}{2\xi_j} \quad (j = 1, 2, \dots, n). \tag{6}$$

In the immediate vicinity of the source the motion in the pipeline will approach the motion in flow past an infinite diaphragm. In this case the stream surfaces are hyperboloids of revolution, whose intersections with the meridional plane are confocal hy-

parabolas

$$\frac{\xi^2}{c^2 \sin^2 \zeta} - \frac{\eta^2}{c^2 \cos^2 \zeta} = 1,$$

where c is half the distance between the foci, and the value of the stream function is given by the expression $\psi^* = 1 - \cos\zeta$. In Fig. 2, I, 8 streamlines are shown, corresponding to values of ψ^* from $\psi^* = 0$ to $\psi^* = 0.5$. This figure also shows the grid corresponding to the finite-difference method, by means of which, in turn, the values of ψ^* were computed. The results obtained by both methods have an accuracy of 5%.

Fig. 2

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

Grid method. The differential equation (2) can be replaced by the difference equation

$$\psi_2^* + \psi_4^* + \left(1 - \frac{h}{2\xi_0}\right)\psi_1^* + \left(1 + \frac{h}{2\xi_0}\right)\psi_3^* - 4\psi_0^* = 0, \tag{7}$$

where ψ_i^* are the values of the function ψ^* at the points ξ_i, η_i (Fig. 3) corresponding to the nodes of a grid of spacing h.

A direct solution of system (7) of n linear equations in n unknowns is very laborious. Therefore the following method of successive approximation is applied. At each node the quantity

$$F_0 = \psi_2^* + \psi_4^* + \left(1 - \frac{h}{2\xi_0}\right)\psi_1^* + \left(1 + \frac{h}{2\xi_0}\right)\psi_3^* - 4\psi_0^*, \tag{8}$$

is determined, which, in the terminology of Southwell and Allen ([2,3]), may be called the "residual." In view of the fact that the values ψ_i^* at the nodes are initially arbitrary, it is clear that the residual for a single node, generally speaking, differs from zero and represents a measure of the error made at the given node.

Figure 4 gives a scheme representing the change in the values of the residuals at neighboring nodes when ψ_0^* receives an increment equal to +1.

The computation consists in the successive application of the scheme given in Fig. 4, beginning with the largest residual in the given region, until the values of all residuals have decreased to a chosen sufficiently small quantity. The value of the function at a certain point is equal to the initial value, taken arbitrarily, plus the algebraic sum of all increments.

Fig. 3

Fig. 4

Fig. 4

Figure 4: Fig. 4

In Fig. 2, II represents the case of a sphere of radius $a = \frac{1}{2}R$, located in a pipeline of radius R, where the flow rate $Q = 128\pi$. For the extreme streamlines (at the wall of the pipeline) the value obtained is $\psi_{\mathrm{b}}^{*} = 64$.

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Note: Figure translations are in progress. See original paper for figures.