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Abstract

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ON THE SPECTRAL REPRESENTATION OF CHANDRASEKHAR' S AXIALLY SYMMETRIC TURBULENCE

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The occurrence of axially symmetric turbulence is connected with the presence of a physically distinguished direction in the problem (for example, the gradient of the temperature field, etc.). Let us consider the case of an incompressible viscous fluid in the presence of a temperature field. Incompressibility, as usual in problems connected with heat conduction, is understood in the sense that only density changes associated with thermal expansion in the temperature field are significant; density changes caused by motion and by changes of pressure are assumed to be insignificant and to play no role.

The equations of the problem, under the stated assumptions and in the presence of a constant vertical temperature gradient, have, as is known, the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\nabla\Omega + \nu\Delta\mathbf{V} + \gamma\lambda T',$$

$$\frac{\partial T'}{\partial t} + \beta(\lambda\mathbf{V}) + (\mathbf{V}\nabla)T' = \chi\Delta T', \tag{1}$$

$$\Delta\Omega = \gamma(\lambda\nabla)T' - \frac{\partial^2}{\partial x_i \partial x_k}(V_{iV}k),$$

where $\lambda = \mathbf{g}/g$; $\beta = (\lambda\nabla)T$; $\gamma = g\alpha$; α is the coefficient of thermal expansion of the fluid; T' is the temperature fluctuation,

$$\Omega = \frac{p}{\rho} + \mathbf{g}(\lambda\mathbf{r}) - \frac{1}{2}\beta\gamma(\lambda\mathbf{r})^2.$$

In the approximation in which the influence of third-order correlation moments is neglected, anisotropic turbulence in our case is described by means of the system of correlation moments:

$$b_{ij} = \overline{v_i v_j'};$$

$$\Lambda_i = \frac{1}{2}\left(\overline{TV_i'} - \overline{V_i T'}\right);$$

$$L_i = \frac{1}{2}\left(\overline{TV_i'} + \overline{V_i T'}\right);$$

$$\Psi = \frac{1}{2}\left(\overline{\Omega T'} + \overline{T\Omega'}\right);$$

$$\Theta = \overline{TT'};$$

$$P_i = \frac{1}{2}\left(\overline{\Omega V_i'} + \overline{V_i \Omega'}\right); \tag{2}$$

$$\Pi_i = \frac{1}{2}\left(\overline{\Omega V_i'} - \overline{V_i \Omega'}\right);$$

$$\Phi = \frac{1}{2}\left(\overline{\Omega T'} - \overline{T\Omega'}\right).$$

For homogeneous turbulence the system of equations for the moments (2) has the form (¹)

$$\begin{gathered}
\frac{\partial b_{ij}}{\partial t} = \gamma(\lambda_i \Delta_j + \lambda_j \Delta_i) + \frac{\partial \Pi_i}{\partial \rho_j} + \frac{\partial \Pi_j}{\partial \rho_i} + 2\nu \Delta b_{ij}; \\
\frac{\partial \Lambda_i}{\partial t} = \frac{\partial \Phi}{\partial \rho_i} + 2\gamma \lambda_i \Theta - \beta \lambda_i b_{ij} + (\chi + \nu)\Delta \Lambda_i; \\
\frac{\partial L_i}{\partial t} = \frac{\partial \Psi}{\partial \rho_i} + (\chi + \nu)\Delta L_i; \qquad \Delta \Phi = \gamma \lambda_j \frac{\partial \Theta}{\partial \rho_j}; \\
\Delta P_i = -\gamma \lambda_j \frac{\partial L_i}{\partial \rho_j}; \qquad \Delta \Pi_i = -\gamma \lambda_j \frac{\partial \Lambda_i}{\partial \rho_j}; \qquad \vec{\rho} = \overrightarrow{MM'}.
\end{gathered} \tag{3}$$

The spectral density corresponding to b_{ij} is defined by the formulas (²)

$$b_{ij}(\vec{\rho}, t) = \int e^{i(\mathbf{p}\rho)} f_{ij}(\mathbf{p}, t)\, d\tau_p; \qquad f_{ij}(\mathbf{p}, t) = \frac{1}{8\pi^3} \int e^{-i(\mathbf{p}\rho)} b_{ij}(\vec{\rho}, t)\, d\tau_{\vec{\rho}}. \tag{4}$$

Analogous formulas are written for the scalar and vector functions of the problem. Let us denote the spectral densities corresponding to the vectors $\Lambda_i, L_i, \Pi_i, P_i$ by $R_i^{(1)}, R_i^{(2)}, C_i^{(2)}, C_i^{(1)}$; and those corresponding to the scalars Φ, Ψ, Θ by S_1, S_2, T.

By virtue of the incompressibility condition for the fluid, $\partial V_i/\partial x_i = 0$, the vector and tensor correlation moments will be solenoidal. An axisymmetric tensor b_{ij} solenoidal in both indices, and a solenoidal axisymmetric vector L_i, have the form (^{1})

$$\begin{gathered} b_{ij} = \varepsilon_{jlm} \frac{\partial}{\partial \rho_i} \left(b_1 \varepsilon_{ims} \rho_s + b_2 \lambda_m \varepsilon_{irs} \lambda_r \rho_s + \frac{1}{\rho} \frac{\partial b_1}{\partial \mu} \rho_m \varepsilon_{irs} \lambda_r \rho_s \right), \\ L_i = \varepsilon_{ijk} \frac{\partial}{\partial \rho_j} \left(L \varepsilon_{krs} \lambda_r \rho_s \right), \end{gathered} \tag{5}$$

where b_1, b_2, L are scalar functions defining b_{ij} and L_i, depending on $\rho = |\vec{\rho}|$ and $\mu = \cos(\widehat{\vec{\lambda} \vec{\rho}})$; $\vec{\lambda}$ is the unit vector defining the axis of symmetry of the anisotropy; ε_{ijk} is the unit pseudotensor of rank three.

The spectral densities corresponding to (5)—the tensor f_{ij} and the vector $R_1^{(2)}$—have the form:

$$\begin{gathered} f_{ij} = - \left[p^2 \delta_{ij} - p_{ip} j \right] f_1 - \\ - \left[p_{ip} j + p^2 \lambda_i \lambda_j + \left((\vec{p} \vec{\lambda})^2 - p^2 \right) \delta_{ij} - \lambda_{jp} i (\vec{p} \vec{\lambda}) - \lambda_{ip} j (\vec{p} \vec{\lambda}) \right] f_2; \\ R_1^{(2)} = \left[\lambda_i p^2 - p_i (\vec{p} \vec{\lambda}) \right] R_2, \end{gathered} \tag{6}$$

where f_1, f_2, R_2 are scalar functions depending on $p = |\mathbf{p}|$ and $\mu_1 = \cos(\widehat{\mathbf{p} \vec{\lambda}})$. From (4), (5), and (6) there follow formulas relating f_i $(i = 1, 2)$, R_2 to b_i and L:

$$b_2(\rho, \mu, t) + \mu \frac{\partial}{\partial \mu} b_1(\rho, \mu, t) = \frac{i}{\rho^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} p \rho \cos \varphi \, e^{i p \rho \cos \varphi} f_2(p, \mu_1, t) \, p^2 \sin \varphi \, dp \, d\varphi \, d\psi;$$

$$b_1(\rho, \mu, t) = \frac{i}{\rho^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} p \rho \cos \varphi \, e^{i p \rho \cos \varphi} f_1(p, \mu_1, t) \, p^2 \sin \varphi \, dp \, d\varphi \, d\psi; \tag{7}$$

$$L(\rho, \mu, t) = \frac{i}{\rho^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} p \rho \cos \varphi \, e^{i p \rho \cos \varphi} R_2(p, \mu_1, t) \, p^2 \sin \varphi \, dp \, d\varphi \, d\psi,$$

where $\varphi = \widehat{(\mathbf{p}\rho)}$; $\mu_1 = \mu\cos\varphi + \sin(\widehat{\vec{\rho}\vec{\lambda}})\sin\varphi\cos\psi$.

Analogous formulas are written for the remaining vectors. It is also easy to write down the inversion formulas.

Substituting (4) and the analogous formulas into (3) and using (6), we obtain the following system of equations for determining f_1, f_2 and the other defining scalars:

$$\begin{gathered}
\frac{\partial f_1}{\partial t} = 2\gamma p^2(\mu_1^2 - 1)R_1 - 2\nu p^2 f_1;\\
\frac{\partial f_2}{\partial t} = -2\gamma R_1 - 2\nu p^2 f_2;\\
\frac{\partial R_1}{\partial t} = -\frac{2\gamma T}{p^2} + \beta f_1 - (\chi+\nu)p^2 R_1;\\
\frac{\partial T}{\partial t} = -2\beta p^2(\mu_1^2 - 1)R_1 - 2\chi p^2 T;\\
p^2 C_1 = i\gamma(\mathbf{p}\vec{\lambda})R_2; \qquad p^2 C_2 = i\gamma(\mathbf{p}\vec{\lambda})R_1;\\
p^2 S_2 = -i\gamma(\mathbf{p}\vec{\lambda})T; \qquad p^2 S_1 = 0.
\end{gathered} \tag{8}$$

The general solution of the system of equations (8) has the form

$$\begin{aligned}
f_1 &= \sum_{i=1}^{4} A_i(p,\mu_1)\alpha_{1i}e^{\Lambda_i t}; & R_1 &= \sum_{i=1}^{4} A_i(p,\mu_1)\alpha_{2i}e^{\Lambda_i t};\\
f_2 &= \sum_{i=1}^{4} A_i(p,\mu_1)\alpha_{4i}e^{\Lambda_i t}; & T &= \sum_{i=1}^{4} A_i(p,\mu_1)\alpha_{3i}e^{\Lambda_i t};
\end{aligned} \tag{9}$$

$$R_2(p,\mu_1,t) = R_2(p,\mu_1,0)e^{-(\chi+\nu)p^2 t},$$

where $\Lambda_1 = -2\nu p^2$, and $\Lambda_2, \Lambda_3, \Lambda_4$ are the roots of the characteristic equation

$$(\Lambda+2\nu p^2)\{\Lambda^3+3(\chi+\nu)p^2\Lambda^2+\Lambda[2\chi(\chi+2\nu)p^4+2\nu(\nu+2\chi)p^4-6\beta\gamma(\mu_1^2-1)]+4\chi\nu(\chi+\nu)p^6-4\beta\gamma(\chi+\nu)p^2(\mu_1^2-1)\} = \tag{10}$$

and the numbers $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}$, up to an arbitrary constant, are equal to:

$$\begin{gathered}
\alpha_{i1} = (\Lambda_i + 2\nu p^2)[\Lambda_i^2 + (3\chi+\nu)p^2\Lambda_i + 2\chi(\chi+\nu)p^4 - 4\beta\gamma(\mu_1^2 - 1)];\\
\alpha_{i2} = -\beta(\Lambda_i + 2\chi p^2)(\Lambda_i + 2\nu p^2); \qquad \alpha_{i3} = -2\beta^2 p^2(\mu_1^2 - 1)(\Lambda_i + 2\nu p^2);\\
\alpha_{i4} = 2(\Lambda_i + 2\chi p^2); \qquad \alpha_{11} = 0; \qquad \alpha_{12} = 0; \qquad \alpha_{13} = 0.
\end{gathered} \tag{11}$$

The arbitrary functions $A_i(p,\mu_1)$ $(i = 1,2,3,4)$ are determined from the initial values of the moment functions. In the same way as in the theory of isotropic turbulence (2), under certain particular assumptions concerning the dependence of the spectral function f_{nn} on p, one obtains well-known solutions for b_{dd} "of source type," found by M. D. Millionshchikov (3), L. G. Loitsyanskii (4), and L. I. Sedov (5). For the case of axially symmetric turbulence one can obtain analogous particular solutions. Thus, for $f_i = C_i$ $(i = 1, 2)$, at the last stages of the degeneration of turbulence,

$$b_2 = \frac{C_2}{(4\nu t)^{5/2}} e^{-\rho^2/8\nu t}. \tag{12}$$

For $f_i = C_i C_n^{1/2}(\mu_1)$, where C_n^p is a Gegenbauer polynomial, we have

$$b_2 = \frac{C_1 C_n^{1/2}(\mu) + C_2 C_{n-1}^{3/2}(\mu)\,\mu}{(4\nu t)^{5/2}} e^{-\rho^2/8\nu t}. \tag{13}$$

Relation (6) is used to calculate the intensity of sound scattering in an axially symmetric turbulent flow. In accordance with A. M. Obukhov' s formula (^{9}), for our case we have:

$$I = \frac{2\pi A_0^2 \omega^2 V}{r^2 c^4} k_i k_j f_{ij} = \frac{2\pi A_0^2 \omega^2 V}{r^2 c^4} \{-\left[p^2 k^2 - (\mathbf{pk})^2\right] f_1 -$$

$$- \left[(\mathbf{kp})^2 + p^2 (\mathbf{k}\lambda)^2 + (\mathbf{p}\lambda)^2 k^2 - 2(\mathbf{k}\lambda)(\mathbf{kp})(\mathbf{p}\lambda) - p^2 k^2\right] f_2 \}, \tag{14}$$

where $\mathbf{p} = k\mathbf{n} - \mathbf{k}$; A_0 and ω are the amplitude and frequency of the incident wave; $\mathbf{n} = \mathbf{r}/r$; $|\mathbf{k}| = \omega/c$; c is the speed of sound.

To calculate f_1 and f_2, specific assumptions (^{7}) are made concerning the form of the dependences $b_1 = b_1(\rho, \mu)$, $b_2 = b_2(\rho, \mu)$.

In conclusion I consider it my duty to express my deep gratitude to V. L. German for proposing the subject of this work and for assistance in carrying it out.

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