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Abstract

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V. Shmidt

A GENERALIZED RIEMANN-HILBERT PROBLEM IN THE CASE OF NEGATIVE INDEX

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In the unit disk T one seeks a solution of the equation

$$\frac{\partial U}{\partial z} - A\overline{U} = 0, \tag{1}$$

satisfying the boundary condition

$$\mathrm{Re}\,[t^{-n}U(t)] = \gamma(t) \tag{2}$$

on the boundary Γ of the disk T. The integer n is called the **index of the problem** (1)–(2). As is known ([1]), this problem reduces to the problem of finding a solution of the equation $\partial_{\overline{z}}w + \overline{A}w + Bw = F$ in the simply connected domain T subject to a boundary condition of the form $\mathrm{Re}[\lambda(t)w] = \gamma$; here $n = \frac{1}{2\pi}\Delta_L \arg\lambda(t)$, $|\lambda(t)| \neq 0$.

In the paper ([1]) a method was indicated which makes it possible to reduce the boundary-value problem (1)–(2), in the case of a simply connected domain and nonnegative index, to an equivalent Fredholm integral equation, without resorting to the general representation of solutions in terms of analytic functions. In the present note the case of negative index is considered, and problem (1) –(2) is reduced to an equivalent nonlinear integral equation, likewise without the aid of the general representation of solutions. It turns out that this integral equation is very convenient for clarifying a number of questions concerning the solvability of the indicated problem.

Considering the function $V(z) = z^{-n}U(z)$, $n < 0$, we obtain for the function $V(z)$ the following boundary-value problem: find a regular solution in T of the equation

$$\frac{\partial V}{\partial z} - A_n(z)\overline{V} = 0, \qquad A_n = \left(\frac{\overline{z}}{z}\right)^n A, \tag{3}$$

satisfying the boundary condition

$$\mathrm{Re}[V(t)] = \gamma(t) \quad (\text{on } L). \tag{4}$$

The index of this problem is equal to zero. Any of its solutions is represented in the form ([1])

$$V(z) = \left[\frac{1}{2\pi i}\int_L \gamma(t)e^{-p_0(t)}\frac{t+z}{t-z}\frac{dt}{t} + iC\right]e^{-ip(z)+\omega(z)}, \tag{5}$$

where

$$\omega(z) \equiv \omega_1(z) + i\omega_2(z) = -\frac{1}{\pi}\iint_T \frac{A_n(\zeta)\overline{V(\zeta)}}{\zeta - z\,V(\zeta)}\,d\xi\,d\eta;$$

$$p(z) \equiv p_1(z) + ip_2(z) = \frac{1}{2\pi i}\int_L \omega_2(t)\frac{t+z}{t-z}\frac{dt}{t};$$

$$p_0(z) \equiv \omega_1(z) + p_2(z);$$

C is a real constant; ω_1, ω_2, p_1 and p_2 are real functions of the variable z. Equation (5) is a nonlinear integral equation for the function $V(z)$, equivalent to the boundary-value problem (3)–(4). Since problem (3)–(4) is always solvable ([1]), the integral equation (5) is also always solvable.

The solution of the original problem (1)–(2) is obtained in the form

$$U(z) = z^n V(z).$$

Since n is a negative number and the functions $e^{-ip(z)}$ and $e^{\omega(z)}$ are nonzero everywhere on $T+L$, the function $U(z)$ will be continuous everywhere on $T+L$ if and only if the function

$$f(z) = \frac{1}{2\pi i}\int_L \gamma(t)e^{-p_0(t)}\frac{t+z}{t-z}\frac{dt}{t} + iC$$

tends to zero at the point $z = 0$ as z^{-n}. Expanding $f(z)$ in a Taylor series in a neighborhood of the point $z = 0$, we obtain the necessary and sufficient conditions for the existence of a solution $U(z)$, continuous everywhere on $T+L$, in the form

$$\int_L \gamma(t)e^{-p_0(t)}t^{-k-1}dt = 0 \qquad (k = 0, \dots, -n-1). \tag{6}$$

Thus, the following theorems hold:

Theorem 1. In the case of a simply connected domain and negative index n, the homogeneous boundary-value problem (1)–(2) ($\gamma(t) \equiv 0$) has no solution continuous everywhere on $T+L$. But there always exists a one-parameter family of solutions having a pole of order $-n$ at one (given) interior point z_0 of the domain T and continuous on $T+L-(z_0)$.

Theorem 2. The nonhomogeneous boundary-value problem (1)–(2) has a solution continuous everywhere on $T+L$ if and only if conditions (6) are satisfied. This solution is unique.

Theorem 3. The nonhomogeneous boundary-value problem (1)–(2) always has a one-parameter family of solutions that have a pole of order $-n$ at one (given) point z_0 inside the domain T.

Theorem 4. The nonhomogeneous boundary-value problem (1)–(2) has a solution with a pole of order q ($q < -n$; q an integer) at a (given) point z_0 inside T if and only if the conditions

$$\int_L \gamma(t) e^{-p_0(t)} t^{-k-1} = 0, \qquad k = 0, \dots, -n-q-1.$$

are satisfied.

This solution is determined uniquely.

In the same way one can prove analogous theorems stating that the generalized Riemann–Hilbert boundary-value problem in the case of negative index n and a simply connected domain always has solutions which have poles at certain interior points z_ν of the domain T, the sum of whose orders is equal to $-n$.

From ([1]) it is known that the nonhomogeneous boundary-value problem (1)–(2) is solvable in the class of continuous functions if and only if the conditions

$$\int_L t^{n+1} w_k(t) \gamma(t)\, ds = 0 \qquad (k = 1, \dots, -2n-1) \tag{7}$$

are satisfied (s is the arc parameter), where w_k is a complete system of solutions of the adjoint homogeneous boundary-value problem:

$$\partial_{\bar z} w + \overline{A} w = 0 \quad \text{in } T; \tag{8}$$

$$\operatorname{Re}\left[t^n t'(s) w(t)\right] = 0 \quad \text{on } L. \tag{9}$$

In the case $n < 0$, the boundary-value problem (8)–(9) has $-2n-1$ linearly independent solutions.

Obviously, the complex conditions (6) are equivalent to $-2n-1$ real equalities of the form

$$\int_L \gamma\chi_j \, ds = 0 \qquad (j = 1, \dots, -2n-1), \tag{10}$$

where

$$\chi_0(t) = e^{-p_0(t)}, \quad \chi_{2k}(t) = e^{-p_0(t)} \operatorname{Re}[it^{-k}], \quad \chi_{2k-1}(t) = e^{-p_0(t)} \operatorname{Im}[it^{-k}] \quad \text{on } L.$$

Since conditions (6) and (7), separately, are necessary and sufficient for the solvability of the boundary-value problem (1)–(2) in the class of continuous functions, it follows from this that the functions $w_k t^{n+1}$ are, on L, linear combinations of the functions χ_j $(j = 1, \dots, -2n-1)$, and we obtain

$$w_0(t) = t^{-n-1} e^{-p_0(t)}, \qquad w_{2k}(t) = t^{-n-1} e^{-p_0(t)} \operatorname{Re}[it^{-k}],$$

$$w_{2k-1}(t) = t^{-n-1} e^{-p_0(t)} \operatorname{Im}[it^{-k}]. \qquad (\text{on } L). \tag{11}$$

The functions (11) are the boundary values of linearly independent solutions of the boundary-value problem (8)–(9). With the aid of the generalized Cauchy formula [1], we can represent these solutions inside the domain T in terms of their boundary values (11). Thus, we have obtained a complete system of solutions of the boundary-value problem (8)–(9), which are represented with the aid of solutions of the original problem (1)–(2).

Moscow State University
named after M. V. Lomonosov

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References

[1] I. N. Vekua, *Matem. sborn.*, 31 (73), no. 2, 217 (1952). [2] N. I. Muskhelishvili, *Singular Integral Equations*, Moscow–Leningrad, 1946.

Note: Figure translations are in progress. See original paper for figures.