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S. A. CHUNIKHIN

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Abstract

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S. A. CHUNIKHIN

SYSTEMS OF NONSPECIAL SUBGROUPS AND

p-NILPOTENCE OF FINITE GROUPS

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§ 1. The discovery by Sylow of his well-known theorem showed that every finite group possesses a certain obligatory set of special (nilpotent) subgroups. In particular, it turned out that to each prime divisor p of the order of a group one may assign a certain special subgroup whose order is divisible by p, in such a way that all the subgroups obtained as a result are pairwise nonisomorphic (“Sylow system of subgroups”).

In connection with this there arises the question of the existence of similar regularities also for nonspecial subgroups of nonspecial groups—special groups, of course, must then be excluded from consideration. In a number of earlier papers ([1−3]) we obtained results in this direction. It turned out that every nonspecial group also possesses a certain obligatory set of nonspecial subgroups, among which there are nonspecial groups all of whose subgroups are already special. The structure of these groups, which we called ([1]) “groups of type S,” was first established by O. Yu. Schmidt ([4]).

In recent years, in a number of papers ([5−9]), the great significance of groups of type S has been found also for other questions of group theory.

However, for describing properties of the set of nonspecial subgroups of a finite group analogous to the properties of the above-mentioned “Sylow system of subgroups,” the use of groups of type S is plainly insufficient, since every nonspecial group which is the direct product of its p-Sylow subgroup and a p-Sylow complement already contains no subgroups of type S whose orders are divisible by p.

In the first part of the present paper the notion of a system of nonspecial subgroups is introduced and studied. It is established that, in order to reveal here the analogy with special subgroups which interests us, groups of type S should be replaced by the broader class of groups of type S_p introduced by us.

This more universal role of groups of type S_p in the questions under consideration is especially evident in Theorems 2 and 5 below (cf. also Theorem 4 of

([3])). As for Theorems 3 and 6 of the present article, they respectively generalize Propositions 3 and 4 of the work of Noboru Ito ([8]).

In the second part of the present paper two criteria for p-nilpotence of finite groups are obtained, as well as a theorem encompassing the results of Wielandt ([10]) and Gaschütz ([11]) concerning the Frattini subgroup.

§ 2. We give the definitions and notation used by us:

$\mathfrak{G}$—some finite group.

$\mathfrak{E}$—the identity group.

p—some prime number.

pd-group (subgroup)—a finite group (subgroup) whose order is divisible by p (see ([2])).

p-special group—a finite group having an invariant p-Sylow subgroup ([12]).

A p-nilpotent group is a finite group having an invariant p-Sylow complement ([8]).

A special group is a finite group all of whose Sylow subgroups are invariant.

A group of type S is a finite nonspecial group all of whose nontrivial subgroups are special ([1]). The principal properties of groups of type S are as follows: their order has the form $q^\beta r^\gamma$, $\beta > 0$, $\gamma > 0$, i.e. is divisible by only two distinct prime numbers; a Sylow subgroup belonging to one of them is invariant, and to the other is noninvariant cyclic; if the invariant subgroup is $\mathfrak{R}$ of order r^γ, and $\mathfrak{R}_1$ of order r^{γ_1} is a maximal normal divisor of the whole group contained in $\mathfrak{R}$, then $r^{\gamma-\gamma_1} \equiv 1 \pmod{q}$; here $\gamma - \gamma_1$ is the least exponent for which such a congruence is possible ([4]).

A p-decomposable group is a finite group that is p-special and p-nilpotent ([3]).

A group of type S_p is a finite nonspecial *pd*-group all of whose nontrivial *pd*-subgroups are special ([2]); there exist only two kinds of groups of type S_p: *pd*-groups of type S; the direct product of a cyclic group of order p and some group of type S, whose order is not divisible by p ([2]).

Π is a nonempty set of some k prime divisors of the order of the group $\mathfrak{G} \neq \mathfrak{E}$.

By a Π-complex of subgroups of the group $\mathfrak{G} \neq \mathfrak{E}$ (briefly, a Π-complex of the group $\mathfrak{G}$) we shall mean any collection Λ of pairwise nonisomorphic subgroups of the group $\mathfrak{G}$, if there exists some one-to-one mapping of Π onto Λ such that the image Λ_p of each $p \in \Pi$ is a *pd*-subgroup (one subgroup whose order is divisible by each $p \in \Pi$ will also be assigned to Π-complexes).

A ΠS-complex, an oriented ΠS-complex, and a ΠS_p-complex are Π-complexes for which all Λ_p will respectively be subgroups of type S, p-special subgroups of type S, and subgroups of type S_p.

An S-complex, an oriented S-complex, and an S_p-complex are, respectively, a ΠS-complex, an oriented ΠS-complex, and a ΠS_p-complex for the case when $\Pi = \Omega$, where Ω is the set of all prime divisors of the order of the group $\mathfrak{G}$.

$\Phi(\mathfrak{G})$ is the Frattini subgroup of the group $\mathfrak{G}$.

We introduce (3) the symbol $T(n_1, n_2)$ as follows: if $(n_1, n_2) = 1$, then set $T(n_1, n_2) = 1$; if, however, $(n_1, n_2) = q_1^{\omega_1} q_2^{\omega_2} \cdots q_l^{\omega_l}$, where $\omega_1 > 0$, $\omega_2 > 0, \dots, \omega_l > 0$, and $q_1, q_2, \dots, q_l$ are distinct prime numbers, then set

$$T(n_1, n_2) = \prod_{i=1}^{l} (q_i^{\omega_i} - 1)(q_i^{\omega_i - 1} - 1) \dots (q_i - 1).$$

We shall say (3) that a prime divisor p of the order of a finite group $\mathfrak{G}$ polarizes the whole positive integer t, if there exists a series

$$\mathfrak{G} = \mathfrak{N}_0 \supset \mathfrak{N}_1 \supset \cdots \supset \mathfrak{N}_\lambda = \mathfrak{E}$$

of normal divisors of $\mathfrak{G}$ such that for every $i = 1, 2, \dots, \lambda$ the number $(T(\mathfrak{N}_{i-1}/\mathfrak{N}_i), t)$ is relatively prime to $d_i = (p, (\mathfrak{G}/\mathfrak{N}_i))$, where $(\mathfrak{N}_{i-1}/\mathfrak{N}_i)$ and $(\mathfrak{G}/\mathfrak{N}_i)$ are the orders, respectively, of the groups $\mathfrak{N}_{i-1}/\mathfrak{N}_i$ and $\mathfrak{G}/\mathfrak{N}_i$.

§ 3. We now present the results we have obtained on the existence, in nonspecial finite groups, of complexes of subgroups of various kinds.

Theorem 1. *A finite group is p-nilpotent if and only if it contains no trivial or nontrivial p-special pd-subgroup of type S.*

Theorem 2. *Every nonspecial pd-group $\mathfrak{G}$ contains a trivial or nontrivial subgroup $\mathfrak{H}$ of type S_p with the following properties: 1) if $\mathfrak{G}$ is not p-nilpotent, then $\mathfrak{H}$ will be a p-special pd-group of type S; 2) if $\mathfrak{G}$ is p-nilpotent but not p-decomposable, then $\mathfrak{H}$ will be a p-nilpotent pd-group of type S; 3) if $\mathfrak{G}$ is p-decomposable, then $\mathfrak{H}$ will be the direct product of a cyclic group of order p and a group of type S whose order is not divisible by p.*

Theorem 2 is a combination and strengthening of Lemma 2 of paper (1), Theorem 2 of paper (2), and Theorem 3 of paper (3).

Theorem 3. *Let $\mathfrak{G}$ be not nilpotent with respect to each $p \in \Pi$. Then $\mathfrak{G}$ has at least one oriented ΠS-set containing k subgroups.*

Theorem 4. *If $\mathfrak{G}$ is indecomposable with respect to each $p \in \Pi$, then it has at least one ΠS-set containing no fewer than*

$$\left[\frac{k+1}{2}\right]$$

subgroups.

Take l prime numbers $q_i > 2$ and then choose l prime numbers p_i so that $q_i \equiv 1 \pmod{p_i}$, $i = 1, 2, \ldots, l$. If $\mathfrak{G}_i$ is a group of type S and of order $p_{iq}i$, then the direct product $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2 \times \cdots \times \mathfrak{G}_l$ will be an indecomposable group for which, evidently, there is no S-set containing more than l subgroups. Since here $k = 2l$, we see that Theorem 4 cannot be strengthened.

Theorem 5. *A nonspecial group* $\mathfrak{G}$ *has at least one* ΠS_p*-set containing no fewer than* $k - 1$ *subgroups.*

Theorem 6. *If* $\mathfrak{G}$ *has no* ΠS_p*-set of nontrivial subgroups containing* k *subgroups, then* $\mathfrak{G}$ *is a* Π*-solvable group.*

For $\Pi = \Omega$, Theorems 3-6 become theorems on S- and S_p-sets.

§ 4. **Theorem 7.** *Let* $\mathfrak{H}$ *and* $\mathfrak{H}_1$ *be normal divisors of* $\mathfrak{G}$*, and let* $\mathfrak{H} \subseteq \mathfrak{H}_1 \cap \Phi(\mathfrak{G})$*. Let* $\mathfrak{H}_1$ *contain a subgroup* $\mathfrak{N}$ *of order* n*, and suppose* $\mathfrak{H}_1 = \mathfrak{N}\mathfrak{H}$*, and all subgroups of order* n *conjugate to* $\mathfrak{N}$ *in* $\mathfrak{G}$ *are conjugate to* $\mathfrak{N}$ *already in* $\mathfrak{H}_1$*. Then* $\mathfrak{N}$ *will be an invariant subgroup in* $\mathfrak{G}$*.*

Putting here $\mathfrak{N} = \mathfrak{P}$, where $\mathfrak{P}$ is an arbitrary Sylow subgroup of $\mathfrak{G}$, we obtain Wielandt' s theorem ([10]). Theorem 10 of Gaschütz' s paper ([11]) also follows easily from our Theorem 7, if for $\mathfrak{N}$ one takes an arbitrary Sylow subgroup of Gaschütz' s group $\mathfrak{M}$.

From Theorem 7 it also follows:

Theorem 8. *Let* $\mathfrak{H}$ *be a normal divisor of* $\mathfrak{G}$*, and let* $\mathfrak{H} \subseteq \Phi(\mathfrak{G})$*. If* $\mathfrak{G}/\mathfrak{H}$ *is* p*-nilpotent, then the group* $\mathfrak{G}$ *itself is also* p*-nilpotent.*

Theorem 8 is an analogue of the above-mentioned theorem of Wielandt ([10]) on special groups and of our theorem on p-special groups (Theorem 16 of paper ([12])).

Theorem 9. *In order that* $\mathfrak{G} \neq \mathfrak{E}$ *be* p*-nilpotent, it is necessary and sufficient that every prime divisor of its order polarize the number* p^ρ*, where* $p^\rho > 1$ *is the highest power of* p *dividing the order of the commutant of* $\mathfrak{G}$*.*

A particular case of Theorem 9 is Frobenius' theorem ([13]), p. 137.

Belorussian Institute of Engineers
of Railway Transport

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Note: Figure translations are in progress. See original paper for figures.