



Soviet-era science, translated into English

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1958

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Abstract

Full Text

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FOUNDATIONS OF A GENERALIZED DIFFERENTIAL-INTEGRAL CALCULUS

(Presented by Academician S. L. Sobolev on 6 VIII 1958)

Definition. If a function $u(x)$ is continuous and infinitely differentiable on some interval, then by a **generalized differential-integral operation of order n** on the function $u(x)$ with respect to the variable x on the interval under consideration we shall mean the series

$$D^n u(x) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1)x^{k-n}}{k! \Gamma(n-k+1) \Gamma(k-n+1)} u^{(k)}(x), \quad (1)$$

provided it converges. Here $\Gamma(z)$ is the gamma function; the differential parameter n is any constant number, real or complex. The application of the operator D^n will be called **generalized differentiation of order n** .

When n is a positive integer, the right-hand side of (1) is identically the derivative of order n ; for negative integral n we obtain the result of successive integration without an additional function containing arbitrary constants.

Additional function. In accordance with definition (1), we have

Fundamental lemma. *Any function having a generalized derivative of order n identically equal to zero has, in the general case, the form*

$$F(n, x) = \sum_{k=1}^{\infty} C_k x^{n-k} \quad (n-k \neq -N; N \text{ natural}). \quad (2)$$

The series (2) is assumed to be convergent and infinitely differentiable; the constant coefficients C_k , within this condition, are arbitrary.

We shall call the function $F(n, x)$ an **additional function**.

Relying on the fundamental lemma, it is easy to prove that two functions differing by an additional function $F(n, x)$ have the same generalized derivative of order n . The converse proposition is also true.

The existence of the additional function follows necessarily from definition (1). Its introduction into the theory of generalized differential-integral calculus distinguishes the proposed version of the theory both from earlier works in this field ^(1,2) and from contemporary works by foreign authors known to us ^(3,6).

Some properties of generalized derivatives. Since, by definition (1), the generalized derivative is represented by a convergent series in the ordinary integral derivatives of the function under consideration, all the usual rules of differential calculus extend to the introduced operation D^n .

Replacing in (1) the function $u(x)$ by the product $u(x) \cdot v(x)$ of functions satisfying the same requirements, we obtain the generalized Leibniz formula

$$D^n u \cdot v = \sum_{k=0}^{\infty} \frac{\Gamma(n+1)}{k! \Gamma(n-k+1)} u^{(k)} D_1^{n-k} v. \quad (3)$$

Let us also indicate two new general properties useful in applications of generalized differential-integral calculus.

By successive application of formulas (1) and (3) one can obtain

$$v D^n u = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{k! \Gamma(n-k+1)} D^{n-k} [v^{(k)} u]. \quad (4)$$

If in formula (3) the right-hand side is an infinitely differentiable convergent series, then, applying generalized differentiation of order m to (3) and regrouping the terms, we find:

$$D^{m+n} u \cdot v = \Gamma(m+n+1) \sum_{k=-\infty}^{\infty} \frac{D^{n+k} u \cdot D^{m-k} v}{\Gamma(n+k+1) \Gamma(m-k+1)} \quad (5)$$

under the conditions: 1) $\text{Re}(m+n+1) > 0$; 2) the series on the right-hand side of (5) converges. The latter requirement can, in particular, be satisfied by choosing one of the differential parameters so that one of the branches of the series terminates.

The indicated rules and relations constitute the core of the working apparatus of generalized differential-integral calculus.

Group of generalized differential-integral operators. If the operation defined by equality (1) exists for arbitrary finite complex values of the parameter γ , then the set $D = \{D^\gamma\}$ of generalized differential-integral operators D^γ forms a group with multiplication operation (7).

Indeed, for any γ_i, γ_j , and γ_k , taken in a specified order, from (1) we have:

$$D^{\gamma_i} D^{\gamma_j} = D^{\gamma_i + \gamma_j} \in D. \quad (I)$$

The associativity law also holds

$$D^{\gamma_i} (D^{\gamma_j + \gamma_k}) = (D^{\gamma_i + \gamma_j}) D^{\gamma_k}, \quad (II)$$

and for every element $D^\gamma \in D$ there exists its inverse

$$D^{-\gamma}D^\gamma = D^0, \quad (\text{III})$$

where D^0 is the identity element

$$D^0u(x) = u(x).$$

It follows from conditions (I)–(III) that the aggregate D is a group. These conditions do not affect the question of the existence of the additional function (2), but if it is set equal to zero, then the group under consideration becomes commutative.

A subgroup (divisor) of the group $\{D^\gamma\}$ is the group $\{D^\beta\}$, where β are arbitrary real numbers, which in turn has as a subgroup (particular case) the group $d = \{D^\alpha\}$ for integral α .

Between the groups of generalized differential operators described above, applied to the field of functions $u(x)$ and $v(x)$, and algebraic groups under the operations of addition and multiplication there exists a correspondence,

which, under certain additional assumptions, becomes an isomorphic mapping serving as a prerequisite for the justification and development of operational calculus.

Analytic semigroups. We shall satisfy the formal requirements of the abstract theory of semigroups ⁽⁸⁾ if, as mandatory, we retain only conditions (I) and (II).

The connection between generalized differentiation and the theory of semigroups is also revealed directly from definition (1). It is not difficult to verify that for $\text{Re}(n) > 0$ there exists the integral representation

$$D^{-n}u(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1}u(t) dt, \quad (6)$$

if the function $u(t)$ is analytic in a neighborhood of the point $t = x$, and if the interval $0 \leq t \leq x$ is contained in the interval of convergence of its Taylor series at this point.

The Riemann-Liouville integral operator in formula (6) defines an analytic semigroup ⁽⁸⁾ associated with the operation of generalized differentiation.

Under other conditions imposed on the choice of the parameter n and the function $u(x)$, generalized differential operators are correlated with semigroups defined by Weyl's operators Γ ^(8,9), used in the theory of Fourier series, the Laplace transform, and Hadamard operators ^(8,10).

The close connection of generalized differentiation with the analytic theory of semigroups makes it possible to draw upon the apparatus of the latter, in particular to specify the classes of functions with which we are dealing. At the same time, the place occupied by generalized differential-integral calculus in modern mathematical analysis becomes clear.

We have no opportunity to report on other applications for lack of space.

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Received
1 X 1957

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Note: Figure translations are in progress. See original paper for figures.

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