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Abstract

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Diffusion of Radiation in a Plane Layer

(Presented by Academician V. A. Ambartsumian, 6 II 1958)

In a note by the author ([1]), the diffusion of radiation in a semi-infinite medium was considered. For this purpose, a probabilistic method proposed earlier by the author ([2,3]) was used. In the present note, the same method is used to consider the diffusion of radiation in a plane layer of finite optical thickness τ_0.

1. We shall assume that in an elementary volume of the medium isotropic scattering of radiation takes place with photon survival probability λ. The calculation of the radiation field in the medium reduces to determining the function $B(\tau, \tau_0)$ from the equation

$$B(\tau, \tau_0) = \frac{\lambda}{2} \int_0^{\tau_0} B(\tau', \tau_0) \operatorname{Ei} |\tau - \tau'| \, d\tau' + g(\tau), \tag{1}$$

where the function $g(\tau)$ characterizes the arrangement of the radiation sources. The solution of equation (1) can be written in the form

$$B(\tau, \tau_0) = g(\tau) + \int_0^{\tau_0} \Gamma(\tau', \tau, \tau_0) g(\tau') \, d\tau', \tag{2}$$

where $\Gamma(\tau', \tau, \tau_0)$ is the resolvent.

The quantity $\Gamma(\tau', \tau, \tau_0) \, d\tau' d\tau$ represents the probability that a photon emitted between the optical depths τ' and $\tau' + d\tau'$ will then be emitted (after diffusion in the medium) between the optical depths τ and $\tau + d\tau$. Taking into account the indicated probabilistic meaning of the resolvent and using the method of adding layers proposed by V. A. Ambartsumian ([4]), one can obtain a comparatively simple equation for determining the resolvent.

Let us add a layer of small optical thickness $\Delta\tau$ to the upper boundary of the medium ($\tau = 0$). It is obvious that

$$\Gamma(\tau' + \Delta\tau, \tau + \Delta\tau, \tau_0 + \Delta\tau) = \Gamma(\tau', \tau, \tau_0) + \Gamma(\tau', 0, \tau_0)\Delta\tau\Gamma(0, \tau, \tau_0),$$

whence

$$\frac{\partial\Gamma}{\partial\tau'} + \frac{\partial\Gamma}{\partial\tau} + \frac{\partial\Gamma}{\partial\tau_0} = \Phi(\tau', \tau_0)\Phi(\tau, \tau_0), \tag{3}$$

where the notation

$$\Gamma(0, \tau, \tau_0) = \Phi(\tau, \tau_0) \tag{4}$$

has been introduced. Adding a layer of small optical thickness $\Delta\tau$ to the lower boundary of the medium ($\tau = \tau_0$), we obtain

$$\Gamma(\tau', \tau, \tau_0 + \Delta\tau) = \Gamma(\tau', \tau, \tau_0) + \Gamma(\tau_0 - \tau', 0, \tau_0)\Delta\tau\Gamma(0, \tau_0 - \tau, \tau_0), \tag{5}$$

whence

$$\frac{\partial\Gamma}{\partial\tau_0} = \Phi(\tau_0 - \tau', \tau_0)\Phi(\tau_0 - \tau, \tau_0). \tag{6}$$

From (3) and (6) it follows that

$$\frac{\partial\Gamma}{\partial\tau'} + \frac{\partial\Gamma}{\partial\tau} = \Phi(\tau', \tau_0)\Phi(\tau, \tau_0) - \Phi(\tau_0 - \tau', \tau_0)\Phi(\tau_0 - \tau, \tau_0). \tag{7}$$

Equation (7) gives (for $\tau' > \tau$):

$$\Gamma(\tau', \tau, \tau_0) = \Phi(\tau' - \tau, \tau_0) + \int_0^\tau \left[\Phi(x + \tau' - \tau, \tau_0)\Phi(x, \tau_0) - \Phi(\tau_0 - x - \tau' + \tau, \tau_0)\Phi(\tau_0 - x, \tau_0)\right] dx. \tag{8}$$

Thus, the function $\Gamma(\tau', \tau, \tau_0)$ of the two variables τ' and τ is expressed in terms of the function $\Phi(\tau, \tau_0)$ of one variable (τ_0 is a parameter).

2. Along with the resolvent $\Gamma(\tau', \tau, \tau_0)$, let us introduce for consideration the probability that a quantum will escape from the medium. Denote by $p(\tau, \eta, \tau_0)\, d\omega$ the probability that a quantum absorbed at optical depth τ will leave the medium through its upper boundary at an angle $\arccos\eta$ to the inward normal within the solid angle $d\omega$. The intensities of the radiation emerging from the medium through the upper and lower boundaries will be, respectively, equal to

$$I(0, \eta, \tau_0) = \frac{4\pi}{\lambda} \int_0^{\tau_0} p(\tau, \eta, \tau_0) g(\tau)\, d\tau,$$

$$I(\tau_0, \eta, \tau_0) = \frac{4\pi}{\lambda} \int_0^{\tau_0} p(\tau_0 - \tau, \eta, \tau_0) g(\tau)\, d\tau. \tag{9}$$

It is easy to see that

$$p(\tau,\eta,\tau_0) = \frac{\lambda}{4\pi}e^{-\tau/\eta} + \frac{\lambda}{4\pi}\int_0^{\tau_0} \Gamma(\tau,\tau',\tau_0)e^{-\tau'/\eta}\,d\tau', \tag{10}$$

$$\Phi(\tau,\tau_0) = 2\pi\int_0^1 p(\tau,\eta,\tau_0)\frac{d\eta}{\eta}. \tag{11}$$

The relations given make it possible to obtain equations for determining $p(\tau,\eta,\tau_0)$.

Multiplying (7) by $e^{-\tau'/\eta}$, integrating with respect to τ' from 0 to τ_0, and using (10) and (11), we find

$$\frac{\partial p}{\partial \tau} = -\frac{1}{\eta}p(\tau,\eta,\tau_0) + 2\pi p(0,\eta,\tau_0)\int_0^1 p(\tau,\eta',\tau_0)\frac{d\eta'}{\eta'} - 2\pi p(\tau_0,\eta,\tau_0)\int_0^1 p(\tau_0-\tau,\eta',\tau_0)\frac{d\eta'}{\eta'}. \tag{12}$$

The quantities $p(0,\eta,\tau_0)$ and $p(\tau_0,\eta,\tau_0)$ entering into (12) may be represented in the form

$$p(0,\eta,\tau_0) = \frac{\lambda}{4\pi}\varphi(\eta,\tau_0), \qquad p(\tau_0,\eta,\tau_0) = \frac{\lambda}{4\pi}\psi(\eta,\tau_0), \tag{13}$$

where $\varphi(\eta,\tau_0)$ and $\psi(\eta,\tau_0)$ are Ambartsumian functions (4).

From comparison of (10) with (2) we see that $p(\tau,\eta,\tau_0) = B(\tau,\tau_0)$ for

$$g(\tau) = \frac{\lambda}{4\pi}e^{-\tau/\eta},$$

i.e., $p(\tau,\eta,\tau_0)$ is determined by the equation

$$p(\tau,\eta,\tau_0) = \frac{\lambda}{4\pi}e^{-\tau/\eta} + \frac{\lambda}{2}\int_0^{\tau_0} p(\tau',\eta,\tau_0)\,\mathrm{Ei}\,|\tau-\tau'|\,d\tau'. \tag{14}$$

Equations (12) and (14) for determining $p(\tau,\eta,\tau_0)$ were obtained earlier by the author (3) in a somewhat different way.

3. From equations (12) and (14), with the aid of (11), we can obtain equations for determining the function $\Phi(\tau,\tau_0)$. From (12) and (11) we find

$$\Phi(\tau,\tau_0) = K(\tau,\tau_0) + \int_0^{\tau} \left[\Phi(\tau',\tau_0)K(\tau-\tau',\tau_0) - \Phi(\tau_0-\tau',\tau_0)L(\tau-\tau',\tau_0)\right] d\tau', \tag{15}$$

$$\Phi(\tau,\tau_0) = L(\tau_0-\tau,\tau_0) - \int_0^{\tau} \left[\Phi(\tau',\tau_0)K(\tau-\tau',\tau_0) - \Phi(\tau_0-\tau',\tau_0)L(\tau-\tau',\tau_0)\right] d\tau', \tag{16}$$

where

$$K(\tau,\tau_0) = \frac{\lambda}{2}\int_0^1 \varphi(\eta,\tau_0)e^{-\tau/\eta}\frac{d\eta}{\eta}, \qquad L(\tau,\tau_0) = \frac{\lambda}{2}\int_0^1 \psi(\eta,\tau_0)e^{-\tau/\eta}\frac{d\eta}{\eta}. \tag{17}$$

From (14) and (11) we obtain

$$\Phi(\tau,\tau_0) = \frac{\lambda}{2}\,\mathrm{Ei}\,\tau + \frac{\lambda}{2}\int_0^{\tau_0} \Phi(\tau',\tau_0)\,\mathrm{Ei}\,|\tau-\tau'|\,d\tau'. \tag{18}$$

Let us note that equation (18) also follows from the integral equation for the resolvent.

Thus, the function $\Phi(\tau,\tau_0)$ can be determined in two ways: 1) from equations (15) and (16), if the functions $\varphi(\eta,\tau_0)$ and $\psi(\eta,\tau_0)$ are known; 2) from equation (18). After $\Phi(\tau,\tau_0)$ has been determined, the functions $\varphi(\eta,\tau_0)$ and $\psi(\eta,\tau_0)$ can be found from the formulas

$$\varphi(\eta,\tau_0) = 1 + \int_0^{\tau_0} \Phi(\tau,\tau_0)e^{-\tau/\eta}\,d\tau, \qquad \psi(\eta,\tau_0) = e^{-\tau_0/\eta} + \int_0^{\tau_0} \Phi(\tau_0-\tau,\tau_0)e^{-\tau/\eta}\,d\tau, \tag{19}$$

which follow from (10) and (13).

4. It follows from what has been said that the function $\Phi(\tau,\tau_0)$ must play an important role in the theory of radiation diffusion. Knowledge of this function makes it possible to determine the radiation field in a plane layer for arbitrary sources of radiation. In many cases $B(\tau,\tau_0)$ is expressed in terms of $\Phi(\tau,\tau_0)$ very simply. Let us give some examples.

1) Let the sources of radiation be distributed uniformly in the medium, i.e. $g(\tau) = 1$. With the aid of (2) and (7) we find

$$B(\tau,\tau_0) = \Psi(\tau_0,\tau_0)\left[\Psi(\tau,\tau_0) + \Psi(\tau_0-\tau,\tau_0) - \Psi(\tau_0,\tau_0)\right], \tag{20}$$

where

$$\Psi(\tau,\tau_0) = 1 + \int_0^{\tau} \Phi(\tau',\tau_0)\,d\tau'. \tag{21}$$

Using (15), (17), and (19), we obtain

$$\Psi(\tau_0, \tau_0) = \frac{1}{1 - \frac{\lambda}{2}(\alpha_0 - \beta_0)}, \tag{22}$$

where α_0 and β_0 are the zeroth moments of the functions $\varphi(\eta, \tau_0)$ and $\psi(\eta, \tau_0)$.

2) Let the medium be illuminated by parallel rays incident on the upper boundary at an angle $\arccos \zeta$ to the normal and producing an illumination, on a surface perpendicular to them, equal to πS. In the present case

$$g(\tau) = \frac{\lambda}{4} S e^{-\tau/\zeta}. \tag{23}$$

Using (2) and (7), we find

$$B(\tau, \zeta, \tau_0) = \frac{\lambda}{4} S \left\{ \varphi(\zeta, \tau_0) e^{-\tau/\zeta} + \int_0^\tau e^{(\tau-\tau')/\zeta} \left[\Phi(\tau', \tau_0)\varphi(\zeta, \tau_0) - \Phi(\tau_0 - \tau', \tau_0)\psi(\zeta, \tau_0) \right] d\tau' \right\}, \tag{24}$$

where the functions $\varphi(\zeta, \tau_0)$ and $\psi(\zeta, \tau_0)$ are defined by formulas (19).

It should be noted that the problem of the diffusion of radiation in a plane layer illuminated by parallel rays has been solved in a number of works (in particular, geophysical ones) by numerical methods. In this approach equation (1), in which the function $g(\tau)$ is given by formula (23), was solved separately for each angle of incidence $\arccos \zeta$. We see that a much simpler way of solving the problem consists in determining $\Phi(\tau, \tau_0)$ from equation (18) and subsequently computing $B(\tau, \zeta, \tau_0)$ by formula (24).

3) Let us find the total probability that a quantum will emerge from the medium. Denoting by $P(\tau, \tau_0)$ the probability that a quantum absorbed at optical depth τ will leave the medium through the upper boundary in all directions, we have

$$P(\tau, \tau_0) = 2\pi \int_0^1 p(\tau, \eta)\, d\eta. \tag{25}$$

Using relations (12), (13), and (21), we obtain

$$P(\tau, \tau_0) = 1 - \left(1 - \frac{\lambda}{2}\alpha_0\right) \Psi(\tau, \tau_0) - \frac{\lambda}{2}\beta_0 \left[\Psi(\tau_0, \tau_0) - \Psi(\tau_0 - \tau, \tau_0) \right]. \tag{26}$$

In the case $\tau_0 = \infty$, formula (26) gives

$$P(\tau) = 1 - \Psi(\tau)\sqrt{1-\lambda}. \tag{27}$$

The results obtained in this note are readily generalized to the case of anisotropic scattering of light in the medium.

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References

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4. V. A. Ambartsumian, DAN, **38**, 257 (1943).

Note: Figure translations are in progress. See original paper for figures.