MATHEMATICS

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Abstract

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ON THE ASYMPTOTIC CHARACTER OF THE FORMULAS OF M. A. LAVRENTIEV

(Presented by Academician M. A. Lavrentiev, 13 V 1958)

In connection with problems on waves and jets, M. A. Lavrentiev developed a theory of conformal mappings of narrow strips ([1–3]). Using geometric methods, M. A. Lavrentiev constructed approximate expressions for functions realizing conformal mappings of strips close to rectilinear ones, and gave accuracy estimates for the values of the boundary derivative. Later, the connection of these formulas with the theory of singular integral equations was established ([7]). The formulas of M. A. Lavrentiev are of great interest for hydrodynamics and make it possible to study approximately the nature of conoidal waves and some problems of flow around bodies ([5], [6]). In the present note it is shown that the expressions obtained by M. A. Lavrentiev are the first terms of certain asymptotic series. The method used in the work makes it possible to extend the results to the case of spatial problems, and also to refine the indicated formulas.

1. Problem A. Consider the problem of mapping the strip $T_w : 0 \le \psi \le 1$ onto the strip $T_z : 0 \le y \le f(x)$ under the condition $w(\infty) = \infty$, where $w(z) = \varphi + i\psi$, $z = x + iy$. The function ψ is harmonic in T_z and satisfies the boundary conditions

$$\begin{aligned} \psi &= 0 \quad \text{for } y = 0, \\ \psi &= 1 \quad \text{for } y = f(x). \end{aligned} \tag{1}$$

Let ε be a certain parameter. Put $\varepsilon x = \xi,\ y = \eta$. The function ψ will satisfy the equation

$$\varepsilon^2 \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0. \tag{2}$$

We shall seek the solution in the form of a series $\psi = \psi_0 + \varepsilon^2 \psi_1 + \varepsilon^4 \psi_2 + \cdots$, where the functions ψ_i satisfy the boundary conditions

$$\begin{aligned} &\psi_i = 0 && \text{for } y = 0, \\ &\psi_0 = 1, \quad \psi_1 = \psi_2 = \cdots = 0 && \text{for } y = f. \end{aligned}$$

In the variables x and y the solution may be represented by the formulas

$$\psi(x,y) = \psi_0(x,y) + \psi_1(x,y) + \dots + \psi_k(x,y) + \dots, \tag{3}$$

where

$$\begin{aligned}
&\psi_0 = a_0 y,\\
&\psi_1 = -\frac{(a_0)''}{3!}\,y^3 + a_1 y,\\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\psi_k = (-1)^k \frac{a^{(2k)}}{(2k+1)!}\,y^{2k+1} + (-1)^{k-1}\frac{a_1^{(2k-2)}}{(2k-1)!}\,y^{2k-1} + \dots + a_k y.
\end{aligned} \tag{4}$$

Here:

$$a_0 = \frac{1}{f}, \qquad a_1 = \frac{a_0''}{3!} f^2, \qquad a_2 = -\frac{a_0^{(4)}}{5!} f^4 + \frac{a_1''}{3!} f^2, \dots \tag{5}$$

$$\dots, \quad a_k = -(-1)^k \frac{a_0^{(2k)}}{(2k+1)!} f^{2k} - (-1)^{k-1} \frac{a_1^{(2k-2)}}{(2k-1)!} f^{2k-2} + \dots + \frac{a_{k-1}''}{3!} f^2.$$

The determination of $\varphi = \operatorname{Re} w$ is reduced to quadratures

$$\varphi(x,y) = \int \frac{dx}{f(x)} + \frac{1}{2!}\frac{f'(x)}{f^2(x)}\,y^2 + \frac{1}{3!}\int \frac{2{f'}^2 - f'' f}{f}\,dx + \dots$$

The solution obtained is of a formal character, and the series (3) diverges in the general case. However, under known restrictions imposed on the function $f(x)$, the solution obtained is asymptotic in character, and then a finite segment of the series (3) can be used for constructing an approximate solution of problem A.

Theorem 1. *In order that the function $\Psi_m(x,y) = \sum_{i=0}^{m} \psi_i$ be an asymptotic solution of problem A, it is necessary and sufficient that the derivatives of the function f satisfy the conditions*

$$f^{(k)}(x) = O(\varepsilon^{l_k}),$$

where $k = 1, 2, \dots, 2m$; l_k are arbitrary positive numbers. In this case the solution of problem A will have the form

$$\psi(x,y) = \Psi_m(x,y) + O(\psi_{m+1}).$$

2. Using formula (3), one can obtain approximate formulas for the values of the modulus of the boundary derivative

$$\left|\frac{dw}{dz}\right| = \frac{1}{f}\left(1 + \frac{1}{3}ff'' - \frac{1}{6}f'^2\right). \tag{6}$$

This formula contains, as a special case, the well-known formula of M. A. Lavrent' ev ([3]). In order to derive it from (6), one should set $w_1 = hw$, $h = O(\varepsilon)$, $f = O(\varepsilon)$, $f' = O(\varepsilon^{1/2})$, $f'' = O(\varepsilon)$. Then, restricting ourselves to terms of order ε^2, we obtain

$$\left|\frac{dw_1}{dz}\right| = \frac{h}{f}\left(1 + \frac{1}{3}ff''\right). \tag{7}$$

3. A more general problem is considered in a completely analogous way.

Problem B. Determine the function $\psi(x,y)$, harmonic in the strip $T_z: \ f_1 \le y \le f_2$ and satisfying the conditions $\psi = u_i(x)$, if $y = f_i$ $(i = 1, 2)$.

The formal solution has the form

$$\psi(x,y) = \psi_0(x,y) + \psi_1(x,y) + \dots, \tag{8}$$

where

$$\psi_0 = \frac{u_1 - u_2}{f_1 - f_2} + \frac{u_2 f_1 - u_1 f_2}{f_1 - f_2};$$

$$\psi_1(x,y) = a_3 y^3 + a_2 y^2 + a_1 y + a_0;$$

$$a_3 = \frac{1}{3!}\{(u_1'' - u_2'')(f_1 - f_2)^2 - (f_1'' - f_2'')(f_1 - f_2)(u_1 - u_2) -$$

$$-2(u_1' - u_2')(f_1' - f_2') + (2f_1' - f_2')^2(u_1 - u_2)\} : (f_1 - f_2)^2;$$

$$a_2 = \frac{1}{2!}\{(u_2'' f_1 + 2u_2' f_1' + u_2 f_1'' - u_1'' f_2 - 2u_1' f_2' - u_1 f_2'')(f_1 - f_2) -$$

$$-(f_1'' - f_2'')(f_1 - f_2)(u_2 f_1 - u_1 f_2) - 2(u_2' f_1 + u_2 f_1' - u_1' f_2 - u_1 f_2')(f_1' - f_2')(f_1 - f_2) +$$

$$+2(f_1' - f_2')^2(u_2 f_1 - u_1 f_2)\} : (f_1 - f_2)^3;$$

$$a_1 = -a_3(f_1^2 + f_1 f_2 + f_2^2) - a_2(f_1 + f_2); \qquad a_0 = a_3 f_2 f_1(f_1 + f_2) + a_2 f_1 f_2(f_1 + f_2)$$

and so on.

There holds a theorem analogous to theorem 1.

Theorem 2. In order that a finite segment of the series (8) be an asymptotic solution of problem B, it is necessary and sufficient that the derivatives $f_i(x)$ and $u_i(x)$ satisfy the conditions

$$f_i^{(l)}(x) = o(\varepsilon^{k_l}), \qquad u_i^{(l)}(x) = o(\varepsilon^{k_l'}),$$

where k_l and k_l' are arbitrary positive numbers.

4. The same method can be used for the construction of asymptotic solutions of spatial problems. In the present note we restrict ourselves to an example of one axisymmetric problem, which is of great importance in the theory of jets.

Problem C. Determine a function $\psi(r, z)$, satisfying inside the strip $T(0 < r < f(z),\ -\infty < z < +\infty,\ 0 \leqslant \theta \leqslant 2\pi)$ the equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0$$

and taking the value 1 on the surface $r = f(z)$.

An asymptotic solution of problem C is given by the function ψ

$$\psi(r, z) = \frac{r^2}{f^2} + \frac{6{f'}^2 - 2ff''}{8f^2}\, r^2 \left(1 - \frac{r^2}{f^2}\right) + \cdots$$

5. The asymptotic solutions of boundary-value problems set forth above may be applied to the construction of approximate solutions of problems in the hydrodynamics of a fluid having a free boundary. Let us consider the simplest problem of steady waves over a smooth horizontal bottom. In dimensionless variables this problem is equivalent to the following.

Determine a function $\psi(x, y)$, harmonic in the strip T_z bounded by the line $y = 0$ and an unknown curve $y = f(x)$. For $y = 0$, $\psi = 0$, and for $y = f(x)$, $\psi = 1$. In addition, for $y = f(x)$ the condition

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + 2\nu f = C; \tag{9}$$

must be satisfied; here ν is the dimensionless parameter $\nu = gh^3/Q^2$; h is the mean depth of the fluid; Q is the discharge; g is the acceleration of gravity.

We choose the constant C so that $\psi = y,\ f = 1$ satisfy equation (9)

$$C = 1 + 2\nu.$$

Suppose that ν is close to 1, and put $\varepsilon = |1-\nu|$. Suppose also that $f-1 = O(\varepsilon)$, $f' = O(\varepsilon^{3/2}),\ f'' = O(\varepsilon^2)$. Then formula (6) can be written as

$$\left|\frac{dw}{dz}\right|^2 = \frac{1}{f^2}\left(1 + \frac{2}{3}ff''\right) + O(\varepsilon^{5/2}). \tag{10}$$

Substituting this estimate into formula (9), we obtain

$$\frac{1}{f^2}\left(1 + \frac{2}{3}ff''\right) + 2\nu f = 1 + 2\nu. \tag{11}$$

Making the substitution $f = 1 + u$, and discarding terms of order higher than the second, we obtain

$$u'' + 3(\nu - 1)u + \frac{9}{2}u^2 = 0. \tag{12}$$

For $\nu < 1$, equation (12) has the solution

$$u = \frac{1-\nu}{\mathrm{ch}^2\sqrt{\frac{3(1-\nu)}{4}}\,x}. \tag{13}$$

This is the well-known equation of the solitary wave [4].

6. Let us consider the unsteady motion of a fluid over a smooth horizontal bottom. An analogous problem under several other assumptions about the nature of the derivatives was considered by Yu. L. Yakimov [8]. In dimensionless variables the problem is formulated as follows.

Determine the function $\varphi(x, y, \tau)$, harmonic in T_z, and the function $f(x, \tau)$ from the conditions:

I. **Boundary conditions:**

a) $\dfrac{\partial\varphi}{\partial n} = 0$ when $y = 0$;

b) $\frac{\partial\varphi}{\partial\tau}+\gamma f+\frac{1}{2}(\nabla\varphi)^2=\text{const}$ when $y=f$;

c) $\frac{\partial f}{\partial\tau}+\frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial y}$ when $y=f$.

II. **Initial conditions:**

a) $f(x,0)=f_1(x)$ when $\tau=0$;

b) $\frac{\partial f}{\partial\tau}=f_2(x)$ when $\tau=0$.

Let us pass, as above, to the variables ξ,η. Then the boundary conditions become the following:

$$\frac{\partial\varphi}{\partial\tau}+\gamma f+\frac{1}{2}\left\{\varepsilon^2\left(\frac{\partial\varphi}{\partial\xi}\right)^2+\left(\frac{\partial\varphi}{\partial\eta}\right)^2\right\}=\text{const},\qquad \frac{\partial f}{\partial\tau}+\varepsilon^2\frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial f}{\partial\xi}=\frac{\partial\varphi}{\partial\eta}, \tag{14}$$

Putting

$$\varphi=\frac{1}{\varepsilon}\varphi_0+\varepsilon\varphi_1+\dots,$$

we find from the transformed Laplace equation (2)

$$\frac{\partial^2\varphi_0}{\partial\eta^2}=0,\qquad \frac{\partial^2\varphi_1}{\partial\eta^2}=-\frac{\partial^2\varphi_0}{\partial\xi^2}. \tag{15}$$

If for φ_0 we take the boundary conditions

$$\frac{\partial\varphi_0}{\partial\eta}=0\quad\text{when } \eta=0;$$

$$\varphi_0=\varphi_\varepsilon\quad\text{when } \eta=f,$$

and for φ_1

$$\frac{\partial\varphi_1}{\partial\eta}=0\quad\text{when } \eta=0;$$

$$\varphi_1=0\quad\text{when } \eta=f,$$

then on the basis of formulas (15) we find

$$\varphi_0 = \varphi_\varepsilon(\xi, \tau),$$

$$\varphi_1 = \frac{1}{2!}\frac{\partial^2 \varphi_1}{\partial \xi^2}(f^2 - y^2).$$

Computing the boundary derivatives of φ after substitution into (14) with the adopted degree of accuracy, we obtain, in the former variables,

$$\frac{\partial \widetilde{\varphi}}{\partial \tau} + \gamma f + \frac{1}{2}\left(\frac{\partial \widetilde{\varphi}}{\partial x}\right)^2 = \text{const}, \qquad \frac{\partial f}{\partial t} + \frac{\partial \widetilde{\varphi}}{\partial x}\frac{\partial f}{\partial x} = -\frac{\partial^2 \widetilde{\varphi}}{\partial x^2} f, \tag{16}$$

where $\widetilde{\varphi} = \varphi[x, f(x,\tau), \tau]$. Having found f and φ from this, one may, by means of the formulas derived, determine the potential throughout the whole region of flow. The system of equations (16) reduces to a quasilinear equation of hyperbolic type.

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Note: Figure translations are in progress. See original paper for figures.