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Abstract

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENCE EQUATIONS

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For a difference equation of the form

$$y(n+k)+\sum_{m=1}^{k} a_m(n) y(n+k-m)=0, \tag{1}$$

whose coefficients satisfy the conditions

$$\lim_{n\to\infty} a_m(n)=a_m, \qquad m=1,2,\dots,k;$$

$$\lambda^k+a_1\lambda^{k-1}+\dots+a_k=(\lambda-\lambda_1)\cdots(\lambda-\lambda_k), \qquad |\lambda_i|\neq|\lambda_j|, \quad i\neq j,$$

Perron' s theorem ($^{1-4}$) is known, asserting that every solution of equation (1) has the form

$$y(n)=C_1 y_1(n)+\dots+C_{ky}k(n), \tag{2}$$

where

$$\lim_{n\to\infty}\frac{y_m(n)}{y_m(n+1)}=\lambda_m.$$

The assertions of Perron' s theorem concerning the asymptotic behavior of the solutions $y_m(n)$ can be substantially refined if one makes certain additional assumptions on the regularity with which the coefficients $a_m(n)$ approach their limiting values.

Theorem 1. *Let the coefficients of equation (1) satisfy the conditions*

$$a_k(n)\neq 0, \qquad n\geq 1; \qquad \lim_{n\to\infty} a_m(n)=a_m, \qquad \sum^{\infty}|a_m(n+1)-a_m(n)|<\infty$$

$$(m = 1, 2, \dots, k);$$

$$\lambda^k + a_1\lambda^{k-1} + \dots + a_k = (\lambda - \lambda_1)\cdots(\lambda - \lambda_k); \qquad \lambda_i \neq \lambda_j, \quad i = j, \quad \lambda_i \neq 0$$

$$(i, j = 1, 2, \dots, k).$$

Put

$$P_n(\lambda) = \lambda^k + a_1(n)\lambda^{k-1} + \dots + a_k(n) = (\lambda - \lambda_1(n))\cdots(\lambda - \lambda_k(n)),$$

$$\lim_{n\to\infty} \lambda_m(n) = \lambda_m.$$

Every solution of equation (1) has the form (2), where

$$y_m(n) \sim \lambda_m^{-1}(1)\cdots\lambda_m^{-1}(n), \qquad n \to \infty.$$

An analogous result is also easily obtained for systems of difference equations. Let a system be given by

$$y(n+1) = A(n)y(n), \tag{3}$$

where

$$y(n) = \{y_1(n), \dots, y_k(n)\}, \qquad A(n) = (a_{ij}(n))_1^k.$$

Denote by $\lambda_m(n)$, $m = 1, 2, \dots, k$, the proper values of the matrix $A(n)$ (for simplicity we shall assume that the $\lambda_m(n)$ are distinct for all $n \geq 1$), and by $t_m(n) = \{t_{m1}(n), \dots, t_{mk}(n)\}$ the corresponding proper vectors, and, finally, $T(n) = (t_{ij}(n))_1^k$.

Theorem 2. Suppose that

$$\sum_{n=1}^{\infty} |a_{ij}(n+1) - a_{ij}(n)| < \infty, \qquad i, j = 1, 2, \dots, k; \qquad \lim_{n\to\infty} A(n) = A.$$

Then, as is not hard to see, there exist limits (under a corresponding choice of the indices $\lambda_m(n)$)

$$\lambda_m = \lim_{n\to\infty} \lambda_m(n), \qquad T = \lim_{n\to\infty} T(n).$$

If $\lambda_m \neq 0$, $m = 1, 2, \dots, k$; $\lambda_m(n) \neq 0$, $n \geq 1$, $m = 1, 2, \dots, k$; $\lambda_i \neq \lambda_j$, $i \neq j$, then every solution of system (3) has the form (2), where

$$y_m(n) \sim \lambda_m^{-1}(1) \cdots \lambda_m^{-1}(n)\, T e_m(n);$$

$$e_m(n) = \{e_{m1}(n), \dots, e_{mk}(n)\}, \qquad \lim_{n\to\infty} e_{ij}(n) = \delta_{ij}.$$

The assertions of Theorems 1 and 2 can be generalized to the case when $\lambda_m(n)$ may increase without bound or tend to zero. We shall give the formulation of such a generalization of Theorem 1.

Put, as above:

$$P_n(\lambda) = \lambda^k + a_1(n)\lambda^{k-1} + \dots + a_k(n) = (\lambda - \lambda_1(n)) \cdots (\lambda - \lambda_k(n)),$$

$$P_{n,m}(\lambda) = \frac{P_n(\lambda)}{\lambda - \lambda_m(n)}.$$

Theorem 3. If the coefficients of equation (1) satisfy the conditions

$$a_k(n) \neq 0, \qquad n \geq 1; \qquad \sum_{n=1}^{\infty} \left| \frac{\lambda_i(n) P_{n+1,j}(\lambda_i(n))}{\lambda_j(n) P_{n+1,j}(\lambda_j(n))} \right| < \infty, \quad i \neq j,\ i, j = 1, 2, \dots, k, \tag{4}$$

then every solution of equation (1) has the form (2), where

$$y_m(n) \sim \mu_m(1) \cdots \mu_m(n), \qquad \mu_m(n) = \frac{P_{n+1,m}(\lambda_m(n+1))}{\lambda_m(n) P_{n+1,m}(\lambda_m(n))}.$$

Let us pass to difference equations of infinite order. Consider the recurrence relation

$$y_n + \sum_{m=1}^{n} a_{m,n} y_{n-m} = 0, \qquad y_0 = 1. \tag{5}$$

Denote

$$P_n(\lambda) = 1 + \sum_{m=1}^{n} a_{m,n} \lambda^m$$

and suppose that there exists a function $\varphi(n)$ possessing the following properties:

1. For $|\lambda| < \varphi(n)$ the function $P_n(\lambda)$ has exactly k simple zeros, say $\lambda_1(n), \dots, \lambda_k(n)$.
2. $$\lim_{n\to\infty} \max_{|z|=1} \left| \frac{P_{n+1}\left(\dfrac{z}{\varphi(n+1)}\right)}{P_n\left(\dfrac{z}{\varphi(n)}\right)} \right| = 1.$$

Theorem 4. Put

$$P_{n,m}(\lambda) = \frac{P_n(\lambda)}{\lambda - \lambda_m(n)}.$$

If

$$\lim_{n\to\infty} \frac{P_{n+1,i}(\lambda_i(n))}{P_{n+1,i}(\lambda_i(n+1))} = 1, \qquad \sum_{n=1}^{\infty} \left| \frac{\lambda_i(n) P_{n+1,j}(\lambda_i(n))}{\lambda_j(n) P_{n+1,j}(\lambda_j(n))} \right| < \infty, \quad i \neq j, \quad i,j = 1,2,\dots,k, \tag{6}$$

then the relation holds ($\varepsilon > 0$ arbitrarily small)

$$y_n = \sum_{m=1}^{k} C_m y_n^{(m)} + O((1+\varepsilon)^n \varphi(1) \cdots \varphi(n)),$$

$$y_n^{(m)} \sim \mu_m(1) \cdots \mu_m(n), \qquad \mu_m(n) = \frac{P_{n+1,m}(\lambda_m(n+1))}{\lambda_m(n) P_{n+1,m}(\lambda_m(n))}.$$

Let us also consider an infinite system of linear equations of the form

$$y_n + \sum_{m=1}^{\infty} a_{m,n} y_{m+n} = 0, \qquad n = 1,2,\dots, \tag{7}$$

and suppose that there exists a function $\varphi(n)$ such that the series

$$P_n(\lambda) = 1 + \sum_{m=1}^{\infty} a_{m,n}\lambda^m$$

converge for $|\lambda| \le \varphi(n)$ and conditions 1 and 2 are satisfied.

Theorem 5. If (6) holds, then every solution of equation (7) satisfying the condition

$$y_n = O((1-\varepsilon)^n \varphi(1) \cdots \varphi(n)), \qquad \varepsilon > 0,$$

has the form $y_n = C_1 y_n^{(1)} + \cdots + C_{ky} n^{(k)}$, where $y_n^{(m)}$ are the same as in Theorem 4.

Application of Theorems 4 and 5 gives interesting refinements of the results obtained in (5) and partly in (6).

Let us now consider the differential equation

$$y^{(k)} + \sum_{m=1}^{k} a_m(x) y^{(k-m)} = 0, \qquad 0 \le x < \infty, \tag{8}$$

whose coefficients are twice continuously differentiable functions. Introduce the notation

$$P(x;\lambda) = \lambda^k + a_1(x)\lambda^{k-1} + \cdots + a_k(x) = (\lambda - \lambda_1(x)) \cdots (\lambda - \lambda_k(x)),$$

$$P_m(x;\lambda) = \frac{P(x;\lambda)}{\lambda - \lambda_m(x)}, \qquad P'_m(x;\lambda) = \frac{\partial}{\partial\lambda} P_m(x;\lambda).$$

Theorem 6. If

$$\int_0^{\infty} \left| \frac{d}{dx} \frac{\lambda'_i(x)}{\lambda_j(x) - \lambda_i(x)} \frac{P'_j(x;\lambda_i(x))}{P_j(x;\lambda_j(x))} \right| dx < \infty, \qquad i,j = 1,2,\ldots,k, \quad i \ne j,$$

then every solution of equation (8) has the form $y = C_1 y_1 + \cdots + C_{ky} k$, where

$$y_m(x) \sim \exp\left\{ \int_0^x \mu_m(t)\,dt \right\}, \qquad \mu_m(x) = \lambda_m(x) - \lambda'_m(x) \frac{P'_m(x;\lambda_m(x))}{P_m(x;\lambda_m(x))}.$$

Consider the integral equation of the form

$$y(x) + \int_0^x a(x, x-t)y(t)\,dt = \delta(x), \qquad 0 \le x < \infty, \tag{9}$$

where $\delta(x)$ is the delta function, and $a(x,t)$ has two continuous derivatives

with respect to x (with respect to t the function $a(x,t)$ may be a generalized function). Denote

$$P(x;\lambda) = 1 + \int_0^x a(x,t)e^{-\lambda t}\,dt$$

and suppose that there exists a function $\varphi(x)$ satisfying the conditions:

1^*. In the half-plane $\operatorname{Re}\lambda > \varphi(x)$ the function $P(x;\lambda)$ has exactly k simple zeros, say $\lambda_m(x)$, $m = 1, 2, \dots, k$.

2^*. There is an α such that $\lambda^\alpha P(x;\lambda) \to \infty$ as $|\operatorname{Im}\lambda| \to \infty$, uniformly in $\operatorname{Re}\lambda$, $\varphi(x) \leqslant \operatorname{Re}\lambda < \infty$.

3^*. Uniformly in λ on any finite segment of the line $\operatorname{Re}\lambda = 1$,

$$\lim_{n\to\infty} \frac{d}{dx} \ln P\left(x; \frac{\lambda}{\varphi(x)}\right) = 0.$$

Theorem 7. Put

$$P_m(x;\lambda) = \frac{P(x;\lambda)}{\lambda - \lambda_m(x)}, \qquad P'_m(x;\lambda) = \frac{\partial}{\partial\lambda} P_m(x;\lambda).$$

If

$$\lim_{x\to\infty} \lambda'_i(x) \frac{P'_i(x;\lambda_i(x))}{P_i(x;\lambda_i(x))} = 0, \qquad \int_0^\infty \left| \frac{d}{dx} \frac{\lambda'_i(x) P'_j(x;\lambda_i(x))}{\lambda_j(x) - \lambda_i(x)\, P_j(x;\lambda_j(x))} \right| dx < \infty, \tag{10}$$

$$i,\ j = 1, 2, \dots, k, \quad i \ne j,$$

then for a solution of equation (9) we have ($\varepsilon > 0$ arbitrarily small)

$$y(x) = \sum_{m=1}^{k} C_m y_m(x) + O\left(\exp\left\{(1+\varepsilon)\int_0^x \varphi(t)\,dt\right\}\right),$$

where

$$y_m(x) \sim \exp\left\{\int_0^x \mu_m(t)\,dt\right\}, \qquad \mu_m(x) = \lambda_m(x) - \lambda'_m(x)\frac{P'_m(x;\lambda_m(x))}{P_m(x;\lambda_m(x))}. \quad (11)$$

Finally, consider an integral equation of the form

$$y(x) + \int_x^\infty a(x, t-x)y(t)\,dt = 0, \qquad 0 \leqslant x < \infty, \qquad (12)$$

where $a(x,t)$, as before, is assumed to have two continuous derivatives with respect to x. Suppose that there exists a $\varphi(x)$ such that the integral entering the formula

$$P(x;\lambda) = 1 + \int_0^\infty a(x,t)e^{-\lambda t}\,dt,$$

converges uniformly in the half-plane $\operatorname{Re}\lambda \geqslant \varphi(x)$ and the conditions $1^*, 2^*, 3^*$ are fulfilled.

Theorem 8. If (10) holds, then the assertion of Theorem 6 is valid.

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Note: Figure translations are in progress. See original paper for figures.