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Abstract

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SOME HOMOLOGICAL INVARIANTS OF EQUIMORPHIC TRANSFORMATIONS (EQUIMORPHISMS)*

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A one-to-one and uniformly continuous in both directions mapping f of a metric space R_1 onto a metric space R_2 is called an **equimorphic transformation** or **equimorphism**. In the present note homological invariants of equimorphic transformations are constructed for a very broad class of metric spaces (geodesic spaces ([1])), including, in particular, all Riemannian manifolds in which the metric is defined as an intrinsic geodesic metric. The significance of this construction follows from the fact that the invariants defined here may turn out to be nontrivial for Riemannian manifolds homeomorphic to Euclidean space. As usual, the construction of homological invariants is based on a certain coefficient domain.

Let us recall the definition of a geodesic space. A complete metric space R is called **geodesic** if, for any two arbitrary points x and y of it, there exists a point z such that

$$\rho(x, z) = \rho(z, y) = \frac{1}{2}\rho(x, y).$$

Definition. Let R be a geodesic space. If Y is an arbitrary continuous chain, then by $d(Y)$ we shall denote the diameter of the carrier of the chain Y. To each natural number k we assign a domain P_k of the space R. A point x belonging to R belongs to the set P_k if and only if there exists such a continuous cycle $Z \sim 0$ from R with carrier containing the point x, that for every chain X with boundary Z the inequality

$$\frac{d(X)}{d(Z) + 1} > k$$

is satisfied.

The subgroup of the group of continuous r-dimensional homologies of the domain P_k, consisting of all those classes which are homologous to zero in R, will be denoted by H_k^r. Since $P_k \supset P_{k+1}$, there is a natural homomorphism φ_k of the group H_{k+1}^r into the group H_k^r; the limit ([2]) of the inverse sequence of homomorphisms thus defined will be denoted by Q^r. It turns out that the group Q^r is an invariant of equimorphic transformations of the geodesic space R.

The proof of invariance is based on the following proposition ([3]). Let f be an equimorphic mapping of the geodesic space R_1 onto the geodesic space R_2. Then for every number $c > 0$ there exist two such numbers $\alpha > 0$, $\beta > 0$, that the inequality

$$\alpha < \frac{\rho(f(x), f(y))}{\rho(x, y)} < \beta$$

holds whenever $\rho(x, y) > c$.

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* The results set forth in this note form part of a dissertation written by me under the supervision of V. A. Efremovich.

We outline the proof of the invariance of the group Q^r. Denote the subsets P_k in the spaces R_1 and R_2 respectively by ${}_1P_k$ and ${}_2P_k$. From the property of equimorphic mappings of geodesic spaces formulated above it follows that for each k there exists an l sufficiently large that ${}_2P_k \supset f({}_1P_l)$ and $f({}_1P_k) \supset {}_2P_l$. From what has been said it follows immediately that the groups Q_1^r and Q_2^r are isomorphic.

As an application, consider the manifold $\mathfrak{F}^n$ defined in the space of variables $x_1, \dots, x_n, z$ by the equation $z = F(x_1, \dots, x_n)$, where $F(x_1, \dots, x_n)$ is a homogeneous function of degree $l > 1$ and $\sum \left(\frac{\partial F}{\partial x_i}\right)^2 = 0$ only at the point $x_1 = x_2 = \dots = x_n = 0$. We note that the manifold $\mathfrak{F}^n$ is homeomorphic to n-dimensional Euclidean space. Let us compute the group $Q^r = Q^r(\mathfrak{F}^n)$, $r \geqslant 1$, of the manifold $\mathfrak{F}^n$, taking as its metric the intrinsic geodesic metric. For this purpose consider the intersection M^{n-1} of the manifold $\mathfrak{F}^n$ with the pair of planes $z^2 = 1$. It turns out that the group $Q^r(\mathfrak{F}^n)$, $r \geqslant 1$, is isomorphic to the r-dimensional homology group of the manifold M^{n-1}.

The proof is based on the following properties of the sets $P_k = P_k(\mathfrak{F}^n)$. Denote by G the set of all points of the manifold $\mathfrak{F}^n$ at which $z \neq 0$. It turns out that there exists a sufficiently large natural number l such that $P_l \subset G$. On the other hand, for every cycle Z^r, $r \geqslant 1$, in G and for an arbitrary natural number k there exists a cycle $\overline{Z}_r$ in P_k such that $Z^r \sim \overline{Z}_r$ in G.

As a concrete example, consider the manifold Π^n_{st}, defined by the equation

$$z = -x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_{s+t}^2, \qquad s + t = n.$$

From what has been proved it follows immediately that Π^n_{st} and $\Pi^n_{s_2t_2}$ are equimorphic if and only if $s_1 = s_2,\ t_1 = t_2$, or $s_1 = t_2,\ s_2 = t_1$.

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Note: Figure translations are in progress. See original paper for figures.