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Abstract

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ON A DIFFERENCE METHOD FOR SOLVING THE POISSON EQUATION WITH AXIAL SYMMETRY

(Presented by Academician S. L. Sobolev on 28 IX 1957)

In note (1) we proposed a new method for constructing difference equations in solving the axially symmetric Dirichlet problem for the Laplace equation by the method of grids. There we also obtained difference equations involving 9 points in the case of a square grid. It has proved possible to apply these equations also to solving the same problem in the case of the Poisson equation

$$\Delta u = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} = \varphi(r, z), \tag{1}$$

where r is the radial coordinate, and z is the coordinate directed along the axis of symmetry.

1°. Suppose it is required to find a function $u(r, z)$ which, in a domain G of the r, z plane bounded by a curve Γ, satisfies equation (1), and on the boundary Γ assumes the prescribed values $u|_\Gamma = f$.

Assume that the solution $u(r, z)$ of the indicated problem and the function $\varphi(r, z)$ possess in the domain G continuous and bounded derivatives up to the order needed by us.

Cover the domain G with a square grid of mesh size h, and denote an arbitrary node by $\alpha_0 = \alpha_0(r_0, z_0)$, and the 8 nodes nearest to it by $\alpha_i = \alpha_i(r_0 + k_i, z_0 + l_i)$ $(i = 1, 2, \ldots, 8)$, where $k_1 = k_3 = k_6 = h$, $k_2 = k_5 = k_8 = -h$, $k_4 = k_7 = 0$, $l_1 = l_2 = 0$, $l_3 = l_4 = l_5 = h$, $l_6 = l_7 = l_8 = -h$. Denote by u_i the value of the function $u(r, z)$ at the node α_i, if $r_0 \neq 0$, and by $u(k_i, z_0 + l_i)$ when $r_0 = 0$. We shall assume, for simplicity, that the boundary Γ of the domain G intersects the straight lines forming the grid only at grid nodes.

In note (1) we obtained the following difference equations:

$$\frac{L_1(u) + L_2(u)}{2h^2(2\sigma_5 + \sigma_8)} = \Delta_h u = 0 \quad \text{for } r_0 \geq h; \tag{2}$$

$$\frac{L_1^{(0)}(u) + L_2^{(0)}(u)}{12h^2} = \Delta_h^{(0)} u = 0 \quad \text{for } r_0 = 0, \tag{3}$$

where it is denoted

$$L_1(u) = 4[(u_1+u_2)\sigma_1 + (u_4+u_7)\sigma_5] + (u_3+u_5+u_6+u_8)\sigma_3 - 20u_0\sigma_0,$$

$$L_2(u) = \frac{1}{2}\frac{h}{r_0}[4(u_1-u_2)\sigma_2 + (u_3+u_6-u_5-u_8)\sigma_4],$$

$$L_1^{(0)}(u) = 7[u(h, z_0+h) + u(h, z_0-h)] - 58u(0, z_0),$$

$$L_2^{(0)}(u) = 34u(h, z_0) + 5[u(0, z_0+h) + u(0, z_0-h)].$$

Here $\sigma_i = \sigma_i(h/r_0)$ are the same as in (1).

On the basis of Taylor' s formula, we represent the values u_i $(i = 1, 2, \dots, 8)$ and $u(k_i, z_0 + l_i)$ $(i = 1, 3, 4, 6, 7)$ of the function $u(r, z)$ in the form of series, arranged in powers of k_i and l_i, with remainder terms in Lagrange form–

Lagrange. Substituting these expansions into (2) and (3), after some calculations we obtain the relations

$$\Delta_h u = (\Delta u)_{\substack{r=r_0\\z=z_0}} + \frac{2h^2}{4!}E(\Delta u) + \frac{2h^4}{6!}F(\Delta u) + R(u), \tag{4}$$

$$\Delta_h^{(0)} u = (\Delta u)_{\substack{r=0\\z=z_0}} + \frac{h^2}{4!}\left(2\Delta^2 u - \frac{\partial^2}{\partial r^2}\Delta u\right)_{\substack{r=0\\z=z_0}} + R^{(0)}(u), \tag{5}$$

where

$$E(\Delta u) = \frac{1}{2\sigma_5+\sigma_3}\left\{\pi_1\Delta^2 u + \pi_2\frac{\partial^2}{\partial r^2}\Delta u + \pi_3\frac{\partial^2}{\partial z^2}\Delta u + \right.$$

$$+\tau_1\left(4\Delta^3 u + 2\frac{\partial^4}{\partial r^4}\Delta u - 7\frac{\partial^2}{\partial r^2}\Delta^2 u\right) +$$

$$\left. +\tau_2\frac{\partial^2}{\partial z^2}\Delta^2 u + \tau_3\frac{\partial^4}{\partial r^2\partial z^2}\Delta u + \tau_4\frac{\partial^4}{\partial z^4}\Delta u\right\}_{\substack{r=r_0\\z=z_0}},$$

$$F(\Delta u) = \frac{1}{2\sigma_5+\sigma_3}\left\{\chi_1\left(4\Delta^3 u - \frac{1}{2}\frac{\partial^4}{\partial r^4}\Delta u - 2\frac{\partial^2}{\partial r^2}\Delta^2 u\right) + \right.$$

$$+\chi_2 \frac{\partial^2}{\partial z^2}\Delta^2 u + \chi_3 \frac{\partial^4}{\partial r^2 \partial z^2}\Delta u + \chi_4 \frac{\partial^4}{\partial z^4}\Delta u +$$

$$+\mu_1 \left(-24\Delta^4 u + 40\frac{\partial^2}{\partial r^2}\Delta^3 u - 11\frac{\partial^4}{\partial r^4}\Delta^2 u - 2\frac{\partial^6}{\partial r^6}\Delta u\right) +$$

$$+\mu_2 \frac{\partial^2}{\partial z^2}\Delta^3 u + \mu_3 \frac{\partial^4}{\partial r^2 \partial z^2}\Delta^2 u + \mu_4 \frac{\partial^4}{\partial z^4}\Delta^2 u +$$

$$+\mu_5 \frac{\partial^6}{\partial r^4 \partial z^2}\Delta u + \mu_6 \frac{\partial^6}{\partial r^2 \partial z^4}\Delta u + \mu_7 \frac{\partial^6}{\partial z^6}\Delta u \Big\}_{\substack{r=r_0\\ z=z_0}},$$

$$\Delta^n u = \Delta(\Delta^{n-1} u);$$

$$\pi_1 = -2\sigma_2 + 6\sigma_3 + 11\sigma_4 - 12\sigma_5, \qquad \pi_2 = 4\sigma_2 - 8\sigma_3 - 16\sigma_4 + 20\sigma_5,$$

$$\pi_3 = 2\sigma_2 - 5\sigma_3 - 11\sigma_4 + 14\sigma_5;$$

$$\tau_1 = -\frac{4}{15} r_0^2 (\sigma_2 - \sigma_3 - \sigma_4 + \sigma_5), \qquad \tau_2 = \frac{1}{15} r_0^2 (38\sigma_2 - 13\sigma_3 - 23\sigma_4 - 2\sigma_5),$$

$$\tau_3 = -\frac{2}{15} r_0^2 (18\sigma_2 + 7\sigma_3 - 3\sigma_4 - 22\sigma_5),$$

$$\tau_4 = -\frac{2}{15} r_0^2 (14\sigma_2 + 11\sigma_3 + \sigma_4 - 26\sigma_5);$$

$$\chi_1 = 2\alpha_1, \qquad \chi_2 = -24\alpha_1 + 2\alpha_2, \qquad \chi_3 = 8\alpha_1 - \alpha_2, \qquad \chi_4 = 24\alpha_1 - 4\alpha_2 + \alpha_3;$$

$$\mu_1 = \beta_1, \qquad \mu_2 = 4(24\beta_1 - \beta_2), \qquad \mu_3 = -120\beta_1 + 7\beta_2,$$

$$\mu_4 = -144\beta_1 + 12\beta_2 - \beta_3, \qquad \mu_5 = 2(11\beta_1 - \beta_2),$$

$$\mu_6 = 2(60\beta_1 - 7\beta_2 + \beta_3), \qquad \mu_7 = 2(48\beta_1 - 6\beta_2 + \beta_3);$$

$$\alpha_1 = \frac{1}{45}(30\sigma_4 - 7\sigma_3 + 16\sigma_5 + \eta_1), \qquad \alpha_2 = \frac{1}{45}(345\sigma_4 + 167\sigma_3 + 4\sigma_5 + \eta_2),$$

$$\alpha_3 = 15\alpha_1 + 12\sigma_3 - 6\sigma_5;$$

$$\beta_1 = \frac{1}{35} r_0^2(-5\alpha_1 + 2\sigma_2 + \sigma_4), \qquad \beta_2 = \frac{1}{5} r_0^2(-\alpha_2 + 10\sigma_4),$$

$$\beta_3 = \frac{1}{3} r_0^2(-\alpha_3 + 15\sigma_4);$$

$$\eta_1 = 10\frac{r_0^2}{h^2}(2\sigma_2 + 13\sigma_3 + 7\sigma_4 - 22\sigma_5),$$

$$\eta_2 = 40\frac{r_0^2}{h^2}(-\sigma_2 + 16\sigma_3 + 10\sigma_4 - 25\sigma_5);$$

$$|R(u)| \leq C_1 h^6 M_8 + C_2 h^6 \frac{1}{r_0} M_7, \qquad |R^{(0)}(u)| \leq C_3 h^4 M_6.$$

Here C_i $(i = 1, 2, 3)$ are fully determined constants independent of the mesh step h and of u; M_i $(i = 6, 7, 8)$ is the upper bound of the moduli of the i-th derivatives of u in the open domain G for $r > 0$ $(i = 7, 8)$ and $r < h$ $(i = 6)$, in which differentiation with respect to r is performed an even number of times.

If in the right-hand sides of (4) and (5) one substitutes for Δu its value $\varphi(r, z)$ and discards the remainder terms $R(u)$ and $R^{(0)}(u)$, we obtain the system of equations

$$\Delta_h u = \varphi(r_0, z_0) + \frac{2h^2}{4!} E(\varphi) + \frac{2h^4}{6!} F(\varphi), \qquad r_0 \geq h; \tag{6}$$

$$\Delta_h^{(0)} u = \varphi(0, z_0) + \frac{h^2}{4!}\left(2\Delta\varphi - \frac{\partial^2}{\partial r^2}\varphi\right)_{\substack{r=0\\ z=z_0}}, \qquad r_0 = 0, \tag{7}$$

from which, for prescribed values of u on the boundary Γ, one can determine approximate values of u at all interior nodes of the mesh of the domain G.

The solution of this system can be carried out either by the iteration method or by the method of successive group elimination of the unknowns with the aid of matrix inversion.

Let us note that (4), as $r_0 \to \infty$, passes into the well-known difference equation constructed from 9 points of the x, y-plane ([2]).

Let us also note that if, instead of the values u_i and $u(k_i, z_0 + l_i)$ in (2) and (3), one substitutes their representations in the form of infinite Taylor series, then for the remainder terms $R(u)$ and $R^{(0)}(u)$ one can obtain, in the case $\Delta u = 0$, exactly the same expressions as were obtained by us in ([1]) in another way.

Finally, let us observe that in deriving relation (4) we used the easily verified identities

$$2\sigma_1 + 2\sigma_5 + \sigma_3 - 5\sigma_0 \equiv 0,$$

$$2\sigma_1 + 2\sigma_2 - \sigma_3 + \sigma_4 - 4\sigma_5 \equiv 0;$$

$$10\sigma_2 + 12\sigma_3 - 25\sigma_4 - 6\sigma_5 + 10\frac{r_0^2}{h^2}(20\sigma_5 - 11\sigma_3 - 4\sigma_2 - 5\sigma_4) \equiv 0,$$

$$10\sigma_3 - 6\sigma_2 + 21\sigma_4 - 28\sigma_5 + 12\frac{r_0^2}{h^2}(2\sigma_2 - \sigma_3 + \sigma_4 - 2\sigma_5) \equiv 0.$$

2°. Sometimes the following difference equation, more accurate than (7), may be useful for the nodes of the axis:

$$\overline{\Delta}_h^{(0)}(u) = (\Delta u)_{\substack{r=0\\ z=z_0}} + \frac{2h^2}{5!}\,\overline{E}^{(0)}(\Delta u) + \frac{2h^4}{6!}\,\overline{F}^{(0)}(\Delta u) + \overline{R}^{(0)}(u), \tag{8}$$

where

$$\overline{\Delta}_h^{(0)} u = \frac{\overline{L}_1^{(0)}(u) + \overline{L}_2^{(0)}(u)}{15h^2},$$

$$\overline{L}_1^{(0)}(u) = 8\left[u(h, z_0 + h) + u(h, z_0 - h)\right] - 77u(0, z_0),$$

$$\overline{L}_2^{(0)}(u) = 48u(h, z_0) - u(2h, z_0) + 7[u(0, z_0 + h) + u(0, z_0 - h)],$$

$$\overline{E}^{(0)}(\Delta u) = \left[4\Delta^2 u + \left(\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial r^2}\right)\Delta u\right]_{\substack{r=0\\ z=z_0}},$$

$$\overline{F}^{(0)}(\Delta u) = \left[\frac{\partial^2}{\partial z^2}\left(\Delta + \frac{1}{r}\frac{\partial}{\partial r} + 3\frac{\partial^2}{\partial z^2}\right)\Delta u\right]_{\substack{r=0\\ z=z_0}},$$

$$\left|\overline{R}^{(0)}(u)\right| \leq C_4 h^6 \overline{M}_8.$$

Here C_4 is a certain constant; $\overline{M}_8$ is an upper bound for the moduli of the 8th derivatives of u in the open domain G for $r < h$, in which differentiation with respect to r is performed an even number of times.

If in equation (8) the remainder term $\overline{R}^{(0)}(u)$ is discarded and Δu is replaced by its value $\varphi(r, z)$, then we obtain an equation analogous to (7):

$$\overline{\Delta}_h^{(0)} u = \varphi(0, z_0) + \frac{2h^2}{5!}\,\overline{E}^{(0)}(\varphi) + \frac{2h^4}{6!}\,\overline{F}^{(0)}(\varphi).$$

Remark. Equation (3) can also be represented in the form

$$\Delta_h^{(0)} u = (\Delta u)_{\substack{r=0\\ z=z_0}} + \frac{h^2}{4!}\,E^{(0)}(\Delta u) + \frac{h^4}{6!}\,F^{(0)}(\Delta u) + R_1^{(0)}(u), \tag{9}$$

where

$$E^{(0)}(\Delta u) = \left(2\Delta^2 u - \frac{\partial^2}{\partial r^2}\Delta u\right)_{\substack{r=0\\ z=z_0}},$$

$$F^{(0)}(\Delta u) = \left[2\Delta^3 u - \frac{3}{2}\frac{\partial^2}{\partial r^2}\Delta^2 u + \left(\frac{7}{3}\frac{\partial^2}{\partial z^2} - \frac{23}{10}\frac{\partial^2}{\partial r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\Delta u\right]_{\substack{r=0\\ z=z_0}},$$

$$\left|R_1^{(0)}(u)\right| \leq C_5 h^4 M_6' + C_6 h^6 \overline{M}_8.$$

Here C_5 and C_6 are certain constants; M_6' is the least upper bound of the moduli of the derivatives $\partial^6 u/\partial r^6$ in the open domain G for $r = 0$; $\overline{M}_8$ is defined above.

Discarding $R_1^{(0)}(u)$ in (9) and then replacing Δu by its value $\varphi(r, z)$, we obtain the difference equation

$$\Delta_h^{(0)} u = \varphi(0, z_0) + \frac{h^2}{4!}\,E^{(0)}(\varphi) + \frac{h^4}{6!}\,F^{(0)}(\varphi). \tag{10}$$

3°. Example. Suppose it is required to find the solution $u(r, z)$ of the equation

$$\Delta u = 2(1 - r^2)(r^2 + 4) + 4(19r^2 - 17)(1 - z)z - 20(1 - z)^2 z^2 \tag{11}$$

inside a cylinder of height $H = 1$ and base radius $R = 1$, satisfying the condition $u = 0$ on its boundary.

The exact solution of this problem has the form

$$u(r, z) = (1 - r^2)^2(1 - z)z + 5(1 - r^2)(1 - z)^2 z^2.$$

Choosing the mesh step $h = 0.25$ and solving the stated problem, we obtain, at the nodes located on the z-axis, the following results: by formulas (6) and (10), which in the present case are exact equations for $u(r, z)$: 0.363284; 0.562503; by the formulas of S. A. Gershgorin [3]: 0.407904; 0.621124; exact values: 0.363281; 0.562500.

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Note: Figure translations are in progress. See original paper for figures.