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Abstract

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A “GLOBAL”SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR THE NAVIER-STOKES EQUATIONS IN THE CASE OF TWO SPACE VARIABLES

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We consider, in the domain Ω of variation of $x = (x_1, x_2)$, the system of Navier–Stokes equations

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + \sum_{k=1}^{2} v_k \mathbf{v}_{x_k} = -\operatorname{grad} p + \mathbf{f}(x,t), \qquad \operatorname{div}\mathbf{v} = 0 \tag{1}$$

for the functions $\mathbf{v} = (v_1(x,t), v_2(x,t))$ and $p(x,t)$, with boundary and initial conditions

$$\mathbf{v}\big|_S = 0, \qquad \mathbf{v}\big|_{t=0} = \mathbf{a}(x) \quad (\operatorname{div}\mathbf{a} = 0). \tag{2}$$

In the paper ([1]) the unique solvability of problem (1)–(2) (in the case of 2 and 3 space variables) was proved for all instants of time (for $t \geqslant 0$) if $\mathbf{f}$ has a potential and the Reynolds number at the initial instant of time is small, and for a sufficiently small time interval $[0, T]$ if these conditions are not satisfied. In addition, Leray ([2]) (and later we, by another method) proved the unique solvability “globally” (i.e. for all $t \geqslant 0$, without any smallness conditions on the Reynolds number) of the Cauchy problem for system (1) in the case of two space variables. The question of the “global” solvability of the boundary-value problem (1)–(2), even for two space variables, raised doubts (see Leray’s detailed investigations ([3]) on this question). We prove that the following theorem holds:

Theorem. *Problem* (1)–(2) *is uniquely solvable “globally” (i.e. for all* $t \geqslant 0$*, for arbitrary values of the Reynolds number at the initial instant of time and for arbitrary* $\mathbf{f}$*), provided only that the integrals*

$$\int_\Omega \mathbf{a}^2\,dx, \qquad \int_\Omega |\mathbf{v}_t(x,0)|^2\,dx, \qquad \int_0^t\!\int_\Omega [\mathbf{f}^2 + (\mathbf{f}_t)^2]\,dx\,dt$$

are finite.

From the results obtained in ([1]) it follows that the whole question of "global" existence is now reduced to obtaining an a priori estimate for the integral

$$\int_0^t \int_\Omega (\mathbf{v}_t)^2 \, dx \, dt + \int_\Omega \sum_{k=1}^2 v_k^4(x,t) \, dx \tag{3}$$

or $\max|\mathbf{v}|$. In view of this, we shall speak here only about a priori estimates of solutions of problem (1)–(2). It is known (see, for example, ([1])) that for solutions of problem (1)–(2) the inequality holds

$$\int_\Omega \mathbf{v}^2(x,t) \, dx + 2\nu \int_0^t \int_\Omega \sum_{k=1}^2 (\mathbf{v}_{x_k})^2 \, dx \, dt \le$$

$$\le \int_\Omega \mathbf{a}^2 \, dx + 2\left(\int_\Omega \mathbf{a}^2 \, dx\right)^{1/2} \int_0^t \left(\int_\Omega \mathbf{f}^2 \, dx\right)^{1/2} dt + 2\left[\int_0^t \left(\int_\Omega \mathbf{f}^2 dx\right)^{1/2} dt\right]^2. \tag{4}$$

Denote

$$\int_\Omega \sum_{k=1}^2 [\mathbf{v}_{x_k}(x,t)]^2 \, dx = \varphi^2(t).$$

It follows from (4) that the estimate of the integral

$$\int_0^t \varphi^2(t) \, dt$$

is known to us. Let

$$\int_0^t \int_\Omega (\mathbf{f}_t)^2 \, dx \, dt < \infty.$$

Differentiate (1) with respect to t, multiply the result scalarly by $\mathbf{v}_t$, and integrate over Ω and $(0,t)$. After simple transformations we arrive at the inequality

$$\frac{1}{2}\int_\Omega |\mathbf{v}_t(x,t)|^2 \, dx \Big|_{t=0}^{t=t} + \nu \int_0^t \int_\Omega \sum_{k=1}^2 (\mathbf{v}_{tx_k})^2 \, dx \, dt + \int_0^t \int_\Omega \sum_{k=1}^2 v_{kt} \mathbf{v}_{x_k} \mathbf{v}_t \, dx \, dt =$$

$$= \int_0^t \int_\Omega \mathbf{f}_t \mathbf{v}_t \, dx \, dt, \tag{5}$$

from which it easily follows that

$$\psi^2(t)\Big|_{t=0}^{t=t} + 2\nu \int_0^t F^2(t)\,dt \le c \int_0^t \varphi(t) \left[\int_\Omega \sum_{k=1}^{2} v_{kt}^4 \, dx \right]^{1/2} dt + \int_0^t \psi(t) b(t)\,dt, \quad (6)$$

where

$$\psi^2(t) = \int_\Omega |\mathbf{v}_t(x,t)|^2 \, dx, \qquad F^2(t) = \int_\Omega \sum_{k=1}^{2} (\mathbf{v}_{tx_k}(x,t))^2 \, dx, \qquad b^2(t) = 2 \int_\Omega \mathbf{f}_t^2(x,t)\,dx,$$

and c here (and below) denotes constants known to us.

We shall now verify that for any finite continuously differentiable function $u(x_1, x_2)$ of two space variables the following inequality holds:

$$\iint u^4(x_1,x_2)\,dx_1\,dx_2 \le 2 \iint u^2(x_1,x_2)\,dx_1\,dx_2 \iint (u_{x_1}^2 + u_{x_2}^2)\,dx_1\,dx_2, \quad (7)$$

where the integration is carried out over the entire space x_1, x_2.

Obviously,

$$u^2(x_1,x_2) = 2 \int_{-\infty}^{x_k} u u_{x_k}\,dx_k, \qquad k = 1, 2,$$

and therefore

$$\max_{x_k} u^2(x_1,x_2) \le 2 \int_{-\infty}^{\infty} |u u_{x_k}|\,dx_k, \qquad k = 1, 2.$$

Therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^4\,dx_1\,dx_2 \le \int_{-\infty}^{\infty} dx_2 \left(\max_{x_1} u^2 \cdot \int_{-\infty}^{\infty} u^2\,dx_1 \right) \le$$

$$\le 2 \int_{-\infty}^{\infty} dx_2 \left(\int_{-\infty}^{\infty} |u u_{x_1}|\,dx_1 \max_{x_2} \int_{-\infty}^{\infty} u^2\,dx_1 \right) \le 4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u u_{x_1}|\,dx_1\,dx_2 \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u u_{x_2}|\,dx_1\,dx_2,$$

and hence inequality (7) follows.

We shall use inequality (7) to estimate $\int_\Omega \sum_{k=1}^{2} v_{kt}\, dx$ in (6). Since v_{kt} are equal to zero on the boundary S, for them, by virtue of (7), we have

$$\left(\int_\infty v_{kt}^4\, dx\right)^{1/2} \leqslant \sqrt{2}\, \psi(t) F(t),$$

and therefore from (6) it follows that

$$\psi^2(t)\Big|_{t=0}^{t=t} + 2\nu \int_0^t F^2(t)\, dt \leqslant C_1 \int_0^t \varphi(t)\psi(t)F(t)\, dt + \int_0^t \psi(t)b(t)\, dt.$$

Hence, in turn, we conclude successively the validity of the inequalities

$$\psi^2(t)\Big|_{t=0}^{t=t} + 2\nu \int_0^t F^2(t)\, dt \leqslant \nu \int_0^t F^2(t)\, dt + \frac{c_1}{4\nu} \int_0^t \varphi^2\psi^2\, dt + \int_0^t \psi b\, dt,$$

$$\psi^2(t) \leqslant c_2 \int_0^t (\varphi^2 + b^2)\psi^2\, dt + c_3, \tag{8}$$

$$\nu \int_0^t F^2(t)\, dt \leqslant c_2 \int_0^t (\varphi^2 + b^2)\psi^2\, dt + c_3. \tag{9}$$

Since the function $\varphi^2(t) + b^2(t)$ is summable on $[0, t]$, it follows from (8) that

$$\psi^2(t) \leqslant c_4,$$

and from (9)

$$\int_0^t F^2(t)\, dt \leqslant c_5.$$

These inequalities give us an a priori estimate of the solutions, even stronger than (3). From the proof given it is clear that neither the dimensions of the domain nor the smoothness of its boundary affect the values of the constants c_k. The latter depend only on the integrals indicated in the theorem.

Let us note that for any finite nonnegative function $u(x_1, x_2)$ of two variables, alongside (7) the following inequality is also valid:

$$\iint u^3\, dx_1 dx_2 \leqslant \frac{9}{8} \iint u\, dx_1 dx_2 \iint \left(u_{x_1}^2 + u_{x_2}^2\right)\, dx_1 dx_2. \tag{10}$$

The proof of inequality (7) given above is analogous to A. O. Gelfond' s proof of inequality (10). I consider it my pleasant duty to thank A. O. Gelfond, who communicated to me the proof of inequality (10).

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CITED LITERATURE

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3. J. Leray, *J. Math. Pures et Appl.*, s. IX, **13**, 331 (1934).

Note: Figure translations are in progress. See original paper for figures.