EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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MATHEMATICS

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EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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1. We consider the first boundary-value problem in the domain $\overline{D}$ $(\eta_0(t) \leq x \leq \eta_{n+1}(t),\ 0 \leq t \leq T)$ for the equation

$$Lu \equiv u_{xx} - u_t - a(x,t)u_x - b(x,t)u(x,t) = -f(x,t) \tag{1}$$

in the case of piecewise continuous and piecewise differentiable functions $a(x,t)$, $b(x,t)$, and $f(x,t)$. By means of a known transformation [1], the general equation of parabolic type is reduced to equation (1). In particular, the heat-conduction equation

$$u_{\bar{t}} = \left[k(\bar{x},\bar{t})u_{\bar{x}}\right]_{\bar{x}} + f(\bar{x},\bar{t}) \qquad (k(\bar{x},\bar{t}) \geq k_0 > 0) \tag{2}$$

is transformed into the form (1) by the change of variables

$$x = \int^{\bar{x}} \frac{d\alpha}{\sqrt{k(\alpha,\bar{t})}}, \qquad t = \bar{t} \tag{3}$$

If $k(\bar{x},\bar{t})$ has a discontinuity of the first kind on some curve C, then on this curve one usually imposes the continuity conditions for $u(\bar{x},\bar{t})$ and for the flux $(-ku_{\bar{x}})$:

$$[u] = 0, \qquad [ku_{\bar{x}}] = 0. \tag{4}$$

In the new variables (3), the conjugation conditions (4) have the analogous form

$$[u] = 0, \qquad [\sqrt{k}\,u_x] = 0. \tag{4'}$$

2. Consider a finite number of mutually nonintersecting curves in pairs in $\overline{D}$, $\{C_i\}$, $i = 0, 1, \ldots, n+1$, given on the interval $0 \leq t \leq T$ by the equations $x = \eta_i(t)$; renumber them so that $\eta_{i_1}(t) < \eta_{i_2}(t)$ for $i_1 < i_2$. Denote by Δ_i, D the following domains:

$$\Delta_i = (\eta_i(t) < x < \eta_{i+1}(t),\ 0 < t \le T), \qquad 0 \le i \le n; \qquad D = \sum_{i=0}^{n} \Delta_i$$

and introduce the definitions needed below:

1) A collection of curves $\{C_i\}$, $0 \le i \le n+1$, belonging to the closed domain $\overline{D}$, forms a class K_γ if: a) each curve C_i $(0 \le i \le n+1)$ is differentiable and the derivative $\eta_i'(t)$ satisfies, on the interval $0 \le t \le T$, a Hölder condition of order γ; b) the curves $\{C_i\}$ are pairwise nonintersecting in $\overline{D}$.
2) A function $\psi(x,t)$ belongs to the class A_γ^χ $(\psi \in A_\gamma^\chi)$ if it is defined in all domains Δ_i $(0 \le i \le n)$ and in each domain Δ_i satisfies a Hölder condition of order $\gamma > 0$ in t and of order $\chi > 0$ in x.

Obviously, $\psi(x,t)$ is a piecewise-continuous function in D, since it has limiting values on the curve C_i $(0 \le i \le n+1)$.

3) The function $u(x,t)$ is a **regular solution of equation** (1) if it satisfies equation (1) in D and the Hölder conditions in $\bar{D}$, and its derivatives u_x, u_{xx}, u_t are functions of some class A_γ^χ.

The present work arose in connection with the study of the convergence of difference methods used for solving equation (2) in the case of discontinuous $k(x,t)$. Therefore we are interested in the regular solution.

3. **Statement of the problem.** It is required to find, in $\bar{D}$, a regular solution of equation (1) satisfying the initial condition

$$u(x,0) = \varphi(x), \tag{5}$$

the boundary conditions

$$u(\eta_0(t),t) = u_1(t), \qquad u(\eta_{n+1}(t),t) = u_2(t) \tag{6}$$

and the conjugation conditions on the n curves C_i

$$u_{\mathrm{r}i} = u_{\mathrm{l}i}, \qquad q_{\mathrm{r}i}(t)(u_x)_{\mathrm{r}i} - r_{\mathrm{r}i}(t)u_{\mathrm{r}i} = q_{\mathrm{l}i}(t)(u_x)_{\mathrm{l}i} - r_{\mathrm{l}i}(t)u_{\mathrm{l}i}$$

or

$$[u]_i = 0, \qquad [qu_x - ru]_i = 0 \quad \text{for } x = \eta_i(t) \quad (1 \le i \le n), \tag{7}$$

where $[u]_i \equiv u_{\mathrm{r}i} - u_{\mathrm{l}i}$, $u_{\mathrm{r}i} = u(\eta_i(t)+0,t)$, $u_{\mathrm{l}i} = u(\eta_i(t)-0,t)$, and so on.

In particular, for equation (2) we have:

$$q_{\mathrm{r}} = \sqrt{k_{\mathrm{r}}}, \qquad q_{\mathrm{l}} = \sqrt{k_{\mathrm{l}}}, \qquad r_{\mathrm{l}} = r_{\mathrm{r}} = 0, \qquad a = 0.5\, k^{-1/2} k_x - \chi_t, \quad b = 0.$$

The proof of existence of a solution of this problem is carried out in several stages: 1) the source function $G(x, t; \xi, \tau)$ of the same problem for the equation $u_t = u_{xx}$ and $r_{\mathrm{l}i} = r_{\mathrm{r}i} = 0$ is constructed; 2) the properties of heat potentials formed with the aid of $G(x, t; \xi, \tau)$ are studied; 3) the solution of the original problem (1), (5)–(7), with the aid of G, is reduced to an integral equation, which is solved by the method of successive approximations.

4. The source function $G(M, P) = G(x, t; \xi, \tau)$ of our problem for the equation $u_t = u_{xx}$ is, for $M \neq P$, a solution of the equations $G_t - G_{xx} = 0$ and $\bar{G}_\tau + \bar{G}_{\xi\xi} = 0$, and when the arguments coincide ($M = P$) it has a singularity of the same type as the fundamental solution

$$G_0(M, P) = G_0(x, \xi; t - \tau) = \left(2\sqrt{\pi(t-\tau)}\right)^{-1} \exp\left[-(x-\xi)^2/4(t-\tau)\right]. \quad (8)$$

In addition, $G(M, P)$ satisfies the boundary conditions $G = 0$, if $M \in C_s$ or $P \in C_s$ ($s = 0, n+1$), and the conjugation conditions

$$[G]_i = 0, \qquad \left[q\,\frac{\partial G}{\partial x}\right]_i = 0 \quad \text{for } M \in C_i(x = \eta_i(t)), \quad 1 \le i \le n; \quad (9)$$

$$\left[\frac{\bar{G}}{q}\right]_i = 0, \qquad \left[\frac{\partial \bar{G}}{\partial \xi} - \eta_i'(\tau)\,\bar{G}\right]_i = 0 \quad \text{for } P \in C_i(\xi = \eta_i(\tau)), \quad 1 \le i \le n. \quad (10)$$

The bar above means that G is considered as a function of the point $P(\xi, \tau)$. Hence it is seen that G is a discontinuous solution of the conjugate heat-conduction equation.

We shall seek the pair of functions $G(M, P)$ and $\bar{G}(M, P)$ in the form

$$G(M, P) = G_0(M, P) + \sum_{i=0}^{n+1} V_i(M, P); \quad (11)$$

$$\overline{G}(M, P) = G_0(M, P) + \sum_{i=1}^{n} \overline{W}_i(M, P) + \overline{V}_0(M, P) + \overline{V}_{n+1}(M, P), \quad (12)$$

where

$$V_i(M,P) = \int_\tau^t G_0(x,\eta_i(\theta),t-\theta)\mu_i(\theta;P)\,d\theta,$$

$$\overline{W}_i(M,P) = 2\int_\tau^t \frac{\partial G_0}{\partial x}(\eta_i(\theta),\xi;\theta-\tau)\overline{\mu}_i(\theta,M)\,d\theta, \tag{13}$$

$$\overline{V}_s(M,P) = \int_\tau^t G_0(\eta_s(\theta),\xi;\theta-\tau)\overline{\mu}_s(\theta;M)\,d\theta \qquad (s=0,n+1).$$

Requiring that $G(M,P)$ satisfy conditions (9), we obtain $n+2$ integral equations for the functions $\mu_i(t;\xi,\tau)$ $(0 \le i \le n+1)$; conditions (10) for $\overline{G}$ give $n+2$ equations for $\overline{\mu}_i(\tau;x,t)$ $(0 \le i \le n+1)$. The proof of the existence of solutions of these two systems of integral equations for $C_i \in K_\gamma$ is carried out by the method of successive approximations.

The identity $\overline{G}(M,P) \equiv G(M,P)$ holds.

5. Consider the potential

$$F_0(x,t) = \iint_{D_t} G_0(x,\xi;t-\tau)f(\xi,\tau)\,d\xi\,d\tau,$$

where $D_t = (\eta_0(\tau) < \xi < \eta_{n+1}(\tau),\ 0 < \tau < t)$.

Lemma 1. The potential $F_0(x,t)$ is a regular solution in $\overline{D}$ of the equation $u_t = u_{xx} + f(x,t)$, satisfying the conjugation conditions

$$[F_{0x}]_i = 0, \qquad [F_{0xx}+f]_i = 0, \qquad [F_{0t}]_i = 0 \quad \text{for } x=\eta_i(t)\ (1 \le i \le n),$$

if the following conditions are fulfilled: 1) $C_i \in K_\gamma$, $0 \le i \le n+1$, $\gamma > 0$; 2) $f(x,t) \in A^1_\gamma$, $f_x(x,t) \in A^\chi_\gamma$ $(\chi > 0,\ \gamma > 0)$; 3) $f(x_s,0)$ for $x_s = \eta_s(0)$, $s = 0, n+1$; $[f(x,0)]_i = 0$ for $x = \eta_i(0)$, $1 \le i \le n$.

6. **Lemma 2.** The potential

$$F(x,t) = \iint_{D_t} G(x,t;\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau$$

is a regular solution in $\overline{D}$ of the equation $u_t = u_{xx} + f(x,t)$, satisfying the boundary conditions $F = 0$ for $x = \eta_s(t)$ $(s = 0, n+1)$ and the conjugation conditions

$$[F]_i = 0, \qquad [qF_x]_i = 0, \qquad [F_{xx}+\eta_i'(t)F_x+f]_i = 0 \quad \text{for } x=\eta_i(t),\ 1 \le i \le n,$$

if conditions 1) and 3) of Lemma 1 are fulfilled and, in addition: 2a) $f \in A^1_\gamma$, where $\gamma > 1/2$; $f_x \in A^\chi_\gamma$, where $\chi > 0$, $\gamma > 0$; 4) the functions $q_{\ell i}(t)$ and $q_{ri}(t)$ are piecewise continuous on the interval $0 \le t \le T$.

7. Since the function $G(x,t;\xi,\tau)$ is discontinuous in the variables (ξ,τ), one may consider two simple-layer potentials

$$V_j^{\mathrm{r,l}}(x,t) = \int_0^t G(x,t;\eta_j(\theta) \pm 0, \theta)\nu(\theta)\, d\theta$$

along a certain curve C_j $(x = \eta_j(t);\ 1 \le j \le n)$.

Lemma 3. The potentials $V_j^{\mathrm{r}}(x,t)$ and $V_j^{\mathrm{l}}(x,t)$ along a certain curve C_j from the class K_γ $(\gamma > 1/2)$ are regular solutions in $\overline{D}$ of the equation $u_t = u_{xx}$, if condition 1) of Lemma 1 is fulfilled (for $\gamma > 1/2$), condition 4) of Lemma 2, and, in addition, $\nu(0) = 0$, while the derivative $\nu'(t)$ is piecewise continuous on the interval $0 \le t \le T$. On the curves C_i $(1 \le i \le n)$ the derivatives $V_{jxx}^{(\mathrm{r,l})}$, $V_{jxxx}^{(\mathrm{r,l})}$, $V_{jt}^{(\mathrm{r,l})}$ satisfy certain conjugation conditions (which, because of their cumbersome form, we do not give here).

8. **Solution of the initial problem.** Let us represent the solution of problem (1), (5)–(7) in the form of the sum $u(x,t) = v(x,t) + \Phi(x,t)$, where

$$\Phi(x,t) = \varphi\left[\frac{x - \eta_i(t)}{\eta_{i+1}(t) - \eta_i(t)}(\eta_{i+1}(0) - \eta_i(0)) + \eta_i(0)\right] + \psi(x,t),$$

if $M(x,t) \in \Delta_i$ $(0 \le i \le n)$. The function $\psi(x,t)$ is chosen so that $\psi(\eta_0(t),t) = u_1(t) - u_2(0)$, $\psi(\eta_{n+1}(t),t) = u_2(t) - u_2(0)$; $\psi = 0$, $\psi_x = 0$ when $M(x,t) \in C_i$ $(1 \le i \le n)$.

The function $v(x,t)$ satisfies the equation $Lv = -\tilde{f}$, $\tilde{f} = f + L\Phi$, the homogeneous initial and boundary conditions, and also the conjugacy conditions

$$[v]_i = 0, \qquad [qv_x - rv]_i = -\nu_i(t) \quad \text{for } x = \eta_i(t), \quad 1 \le i \le n,$$

where

$$\nu_i(t) = \frac{1}{q_{\eta_i}(t)}[q\Phi_x - r\Phi]_i.$$

Green' s formula gives the equation for $v(x,t)$:

$$v(x,t) = \iint_{\Pi_t} G(x,t;\xi,\tau)\left[a(\xi,\tau)\frac{\partial v}{\partial \xi}(\xi,\tau) + b(\xi,\tau)v(\xi,\tau) + \tilde{f}(\xi,\tau)\right] d\xi\, d\tau$$
$$-\sum_{i=1}^{n}\int_0^t \left[G(x,t;\eta_i(\theta)+0,\theta)\,\frac{[r]_i}{q_{\eta_i}(\theta)}\,v(\eta_i(\theta),\theta) - G(x,t;\eta_i(\theta)+0,\theta)\,\nu_i(\theta)\right] d\theta. \tag{14}$$

9. With the aid of equation (14) and Lemmas 2 and 3 one proves the following:

Existence and uniqueness theorem. There exists, and moreover is unique, a solution of the initial problem (1), (5)–(7), defined and regular in the closed domain $\overline{\Pi}$, if the following conditions are satisfied:

1) The curves $\{C_i\}$ $(0 \le i \le n+1)$ form the class K_γ, with $\gamma > 1/2$.
2) $f \in A^1_\gamma$, where $\gamma > 1/2$; $f_x \in A^\chi_\gamma$, where $\chi > 0$, $\gamma > 0$; $a \in A^1_\gamma$, $b \in A^1_\gamma$, where $\gamma > 1/2$;
   $a_x \in A^\chi_\gamma$, $b_x \in A^\chi_\gamma$, where $\chi > 0$, $\gamma > 0$.
3) The function $\varphi(x)$ on the interval $\eta_0(0) \le x \le \eta_{n+1}(0)$ is continuous and has piecewise-continuous derivatives $\varphi'(x)$, $\varphi''(x)$, $\varphi'''(x)$, and $\varphi'''(x)$ satisfies, on each of the intervals $\eta_i(0) < x < \eta_{i+1}(0)$ $(0 \le i \le n)$, a Hölder condition.
4) The functions $u_1(t)$, $u_2(t)$ have derivatives $u_1'(t)$, $u_2'(t)$, satisfying on the interval $0 \le t \le T$ a Hölder condition of order $\gamma > 1/2$.
5) The functions $q_{\lambda i}(t)$, $r_{\lambda i}(t)$, $q_{\pi i}(t)$, $r_{\pi i}(t)$ $(1 \le i \le n)$ have piecewise-continuous first derivatives on the interval $0 \le t \le T$.
6) The compatibility conditions hold:

$$u_1(0) = \varphi(\eta_0(0)); \qquad u_2(0) = \varphi(\eta_{n+1}(0)); \qquad [\varphi]_i = 0; \qquad [q\varphi' - r\varphi]_i = 0;$$

$$[\varphi'' + (a + \eta_i')\varphi' + b\varphi + f]_i = 0 \quad \text{for } t = 0, \quad x = \eta_i(0) \quad (1 \le i \le n);$$

$$u_1'(0) = (\varphi'' + (a + \eta_0')\varphi' + b\varphi + f) \quad \text{for } t = 0, \quad x = \eta_0(0);$$

$$u_2'(0) = (\varphi'' + (a + \eta_{n+1}')\varphi' + b\varphi + f) \quad \text{for } t = 0, \quad x = \eta_{n+1}(0).$$

10. The method used by us makes it possible to prove an analogous theorem also for a number of other problems, for example:

1) in the case of boundary conditions of the form

$$\alpha_s(t)u_x(x_s, t) + \beta_s(t)u(x_s, t) = u_s(t), \quad \text{where } x_s = \eta_s(t) \quad (s = 0, n+1);$$

2) in the case of conjugacy conditions of the form $[pu]_i = 0$, $[qu_x - ru]_i = 0$ for $x = \eta_i(t)$ $(1 \le i \le n)$, etc.

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CITED LITERATURE

1. M. Gevrey, *J. Math.*, **9**, fasc. IV (1913).

Note: Figure translations are in progress. See original paper for figures.