SOME QUESTIONS ON INFINITESIMAL BENDINGS OF SURFACES

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Abstract

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MATHEMATICS

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SOME QUESTIONS ON INFINITESIMAL BENDINGS OF SURFACES

(Presented by Academician I. N. Vekua, 28 V 1958)

In the present paper, which is a further development of the author' s work (^{1}), a number of questions are considered concerning the rigidity of a surface of positive curvature subject to bushing constraints of the form

$$\vec{U}\vec{\nu} = 0 \quad \text{on } L, \tag{1}$$

where $\vec{\nu}$ is the normal of the bushing; L is the aggregate of the contours of the holes of the surface; $\vec{U}$ is the vector of an infinitesimal bending satisfying the equation

$$dU\,dr = 0, \tag{2}$$

$\mathbf{r}$ is the radius-vector of the surface.

If the bushings are nonrigid, then the bushing constraints have the form

$$\vec{U}\vec{\nu} = \gamma \quad \text{on } L. \tag{3}$$

At each point of L we can consider two trihedra $\mathbf{s}, \mathbf{m}, \mathbf{b}$ and $\mathbf{s}, \mathbf{l}, \mathbf{n}$, where $\mathbf{s}$ is the unit tangent to L; $\mathbf{m}$ and $\mathbf{b}$ are the unit vectors of the principal normal and binormal of L; $\mathbf{l}$ is the tangential normal to L; $\mathbf{n}$ is the external normal of the surface. Let τ be the angle between $\vec{\nu}$ and $\mathbf{b}$, which we assume to be constant along each contour of the surface; θ is the angle between $\mathbf{m}$ and $\mathbf{n}$, measured from $\mathbf{m}$ in the clockwise direction.

The paper also considers the question of rigidity for certain classes of closed nonconvex surfaces.

1. Consider a regular surface of positive curvature. Let it be referred to a conjugate isometric system of coordinates. Then equation (2) can be written in the complex form (^{4}):

$$\frac{\partial W}{\partial \bar{z}} + B\overline{W} = 0 \quad \left(\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right), \tag{4}$$

where

$$W = (a\sqrt{K})^{-1/2} U r_{\bar{z}}; \qquad a = a_{11}a_{22} - a_{12}^2; \qquad K \text{ is the Gaussian curvature;}$$

$$B = \frac{1}{4}\left(\Gamma_{22}^1 - \Gamma_{11}^1 + 2\Gamma_{12}^2\right) + \frac{i}{4}\left(\Gamma_{22}^2 - \Gamma_{11}^2 - 2\Gamma_{12}^1\right).$$

Equations (3) and (1) reduce to the form

$$\vec{U}\vec{\nu} \equiv \operatorname{Re}\left[\alpha\frac{\partial W}{\partial z} + \beta W\right] = \gamma \quad \text{on } L; \tag{5}$$

$$\vec{U}\vec{\nu} \equiv \operatorname{Re}\left[\alpha\frac{\partial W}{\partial z} + \beta W\right] = 0 \quad \text{on } L, \tag{5^0}$$

where

$$\alpha = \frac{\sin(\theta - \tau)}{\sqrt[4]{K}}, \qquad \beta = \frac{\sin(\theta - \tau)}{K\sqrt{a}} \frac{\partial\sqrt{Ka\sqrt{K}}}{\partial z} - \sqrt{a\sqrt{K}}\cos(\theta - \tau)\frac{d\bar{z}}{dl}. \tag{6}$$

We assume that $\sin(\theta - \tau) \neq 0$ (on L), i.e., the collars are nowhere orthogonal to the surface along the contours. Obviously,

$$\varkappa = \frac{1}{2\pi}[\arg\overline{\alpha}]_L = 0. \tag{7}$$

Let l and l^* be the numbers of linearly independent solutions of the homogeneous problem (4), (5^0) and of the adjoint homogeneous problem for the equation $\partial_{\bar{z}}V - \overline{B}\,\overline{V} = 0$. In view of the work (^{2}) and equality (7), we have*

$$l - l^* = 3(1 - m), \tag{8}$$

where $m + 1$ is the connectivity of the surface.

Therefore, for a simply connected surface, under the homogeneous condition (5^0), equation (4) always has $l \geq 3$ solutions, i.e., a surface with one contour under the collar constraint (1) always admits at least three bendings, including motions. (In the works (^{1,3}) the author proved that in some cases all these bendings are trivial.) Hence the following holds:

Theorem 1. *Let S be an ovaloid with one plane contour L. If L is not a circle, then under the collar constraint (1), where $\tau \neq 0,\ \pi/2$ and $\sin(\theta-\tau) \neq 0$ (on L), the surface S will always admit at least three nontrivial bendings.*

This theorem follows from the fact that in this case S admits not a single trivial motion.

Remark 1. If $\tau \neq \text{const}$ along L, then S admits at least one nontrivial bending.

Let $\vec{\gamma} \equiv \mathbf{n}$; in view of Remark 1, the following holds:

Theorem 2. *S will be rigid under the collar constraint $\mathbf{Un} = 0$ (on L) if and only if L is a circle and $\theta = \text{const}$ along L.*

In the case when L is a circle and $\theta = \text{const}$ along L, under the additional condition

$$\operatorname{arc\,tg} \frac{R}{h} \leq \frac{5\pi}{2} - 2\theta$$

S will always be rigid (R is the radius of the circle L; h is the distance from the point P (the point of intersection of the axis of rotation of the circle L with the surface) to the plane of the contour L).

The second part of this theorem follows directly from the work ([1]), if one takes into account that in this case $\tau = \theta - \pi/2$.

Consider an ovaloid S with $m+1$ circular holes, on which nowhere orthogonal nonhomogeneous collar constraints (3) are imposed. In view of equality (8), the following will hold:

Theorem 3. *If, under homogeneous collar constraints (1), S is rigid, then under nonhomogeneous collar constraints (3), S admits nontrivial infinitesimal bendings if and only if the function γ satisfies l^* integral conditions, where: 1) $l^* = 3m-3$, if $m > 1$; 2) $l^* = 2$, if $m = 1$, $\tau \equiv \pi/2$ and the planes of the contours are parallel; 3) $l^* = 1$, if $m = 1$, $\tau \equiv \pi/2$ and the planes of the contours are parallel, or $m = 0$, $\tau = \pi/2$.*

In the case when $m = 0$, $\tau \neq \pi/2$ or $m = 1$ and the planes of the contours are not parallel, the ovaloid always admits nontrivial infinitesimal bendings under nonhomogeneous collar constraints (3) with any right-hand side.

2. Consider closed nonconvex surfaces of revolution. Let S be a closed surface of revolution consisting of two spherical segments S_1, S_2 and a cylinder T, whose meridians are expressed, respectively:

$$\begin{aligned}
\rho_1(z) &= \sqrt{R_1^2-(z-z_1)^2}, && z_0 \leqslant z \leqslant R_1+z_1;\\
\rho_2(z) &= \sqrt{R_2^2-(z+z_2)^2}, && -(R_2+z_2) \leqslant z \leqslant -z_0;\\
\rho_0 &= \sqrt{R_1^2-h_1^2} = \sqrt{R_2^2-h_2^2}, && -z_0 \leqslant z \leqslant z_0
\end{aligned}$$

$$(h_1 = z_1 - z_0, \quad h_2 = z_2 - z_0, \quad z_0, z_1, z_2 \geqslant 0).$$

S_1 and T are glued along the circle L_1 with center $z = z_0$ (on the z-axis) and radius $\rho_1(z_0) = \rho_0$, and T and S_2 are glued along the circle L_2 with center $z = -z_0$ (on the z-axis) and radius $\rho_2(-z_0) = \rho_0$.

Theorem 4. $S(= S_1 + T + S_2)$ admits nontrivial bendings if and only if one of the following conditions is satisfied:

1)

$$h_1 = \frac{R_1}{k}, \qquad h_2 = \frac{R_2}{k} \quad (k = 2, 3, ...);$$

2)

$$z_0 = \frac{1}{k}\,\frac{R_1^2-h_1^2}{2}\left(\frac{1}{kh_1-R_1}+\frac{1}{kh_2-R_2}\right) \quad \text{for} \quad k \geqslant \max\left(\left[\frac{R_1}{h_1}\right], \left[\frac{R_2}{h_2}\right]\right)+1$$

or

$$\left[\frac{R_1+R_2}{h_1+h_2}\right] \geqslant k \geqslant \min\left(\left[\frac{R_1}{h_1}\right], \left[\frac{R_2}{h_2}\right]\right)+1;$$

3)

$$z_0 \equiv 0, \qquad \frac{h_1+h_2}{R_1+R_2} = \frac{1}{k} \quad (k = 2, 3, ...).$$

In the case when $\rho_1(z) \leqslant \rho_0$ and $\rho_2(z) \leqslant \rho_0$, obviously, S will always be rigid. If S_1 and S_2 are concave into the cylinder T, then the closed surface obtained is likewise always rigid.

Let us consider one more class of closed surfaces of revolution $\{S\}$; S consists of S_1 and S_2, whose meridians are expressed respectively by $\rho_1(z) = c_1(a-z)^\alpha$ $(0 \leqslant z \leqslant a,\ 0 < \alpha < 1)$, $\rho_2(z) = c_2\sqrt{R^2-(z+h)^2}$ $(-R-h \leqslant z \leqslant 0)$. S_1 and S_2 are glued along the circle L with center $z = 0$ (on the z-axis) and radius $\rho_1(0) = c_1 a^\alpha$, where $c_1 a^\alpha = c_2\sqrt{R^2-h^2}$.

Theorem 5. $S(= S_1 + S_2)$ admits nontrivial bendings if and only if

$$h = \frac{c_1^2}{c_2 k^2}\,a^{2\alpha-1}\left[\frac{1}{2}+\frac{1}{2}\sqrt{4\alpha(1-\alpha)(k^2-1)+1}+(k^2-1)\alpha\right]+\frac{1}{k}.$$

In particular, for $\alpha = 1/2$ we shall have

$$h = \frac{c_1^2}{2c_2}\left(1 + \frac{1}{k}\right) + \frac{1}{k}.$$

Hence, obviously, in the case when $h \leqslant \frac{c_1^2}{2c_2}$ or $h > \frac{3c_1^2+2c_2}{4c_2}$, S will always be rigid.

Remark 2. Theorems 4 and 5 show that, in the case of gluing, non-rigidity is a comparatively rare case.

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* In the work (^{2}), the formulation of Corollary 4 for the boundary-value problem (5) must read: $l - l^* = \varkappa - \varkappa^* = 2\varkappa - 3(m-1)$. This was pointed out by I. I. Danilyuk.

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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