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Abstract

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MECHANICS

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QUESTIONS ON THE GEOMETRY OF COMPLEX KINEMATIC CHAINS

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We shall call a kinematic chain **closed** if, for any choice of two kinematic pairs P_j, P_k of the chain, there is at least one simple closed contour C, belonging to the same chain, that includes both selected pairs. Consider a closed chain $S \equiv S_0 \dots S_n$. We shall call the simple closed contours of some system belonging to S **independent** if each of them includes at least one kinematic pair that belongs to none of the other contours of the system. If λ is the multiplicity of some link of the kinematic chain under consideration, $n^{(\lambda)}$ is the number of links of multiplicity λ, and $\lambda_{\max}$ is the greatest of the multiplicities of the links of the chain, then the number c of independent simple closed contours belonging to S is determined by the formula

$$c = 1 + \frac{1}{2} \sum_{\lambda=2}^{\lambda_{\max}} (\lambda - 2) n^{(\lambda)}. \tag{1}$$

Indeed, since the chain is assumed to be closed, it includes at least one simple closed contour. If C_1 is one of these contours, then the first term on the right-hand side of equation (1) will correspond to it. If the chain is complex, then among the links of the contour C_1 there will be found at least one link carrying at least one pair distinct from all the pairs of the contour C_1. In fact, there will be at least two such links, since otherwise the chain would not be closed. Any of the newly singled-out kinematic pairs determines at least one simple closed contour distinct from C_1. Let C_2 be one of the newly adjoined contours. C_1 and C_2 are independent of one another. If the kinematic chain under consideration coincides with the system of two simple closed contours C_1, C_2 that we have constructed, then formula (1) is valid. If, however, among the links of the system $C_1 C_2$ there is a link carrying a pair distinct from those singled out earlier, then the preceding arguments can be continued until all pairs entering the kinematic chain under consideration have been exhausted. Thus, the validity of formula (1) is established.

If $1+n$ is the number of links, and p is the number of pairs of the system S, then

$$\sum_{\lambda=2}^{\lambda_{\max}} \lambda n^{(\lambda)} = 2p. \tag{2}$$

Further,

$$\sum_{\lambda=2}^{\lambda_{\max}} n^{(\lambda)} = 1+n. \tag{3}$$

Thus relation (1) takes the form

$$c = p - n. \tag{4}$$

Relation (4) is one of the basic relations determining the structure of closed kinematic chains.

Remark. Relation (4), like (1), also remains valid in the general case of chains closed in the usual sense of this term.

Let p_ρ be the number of kinematic pairs of rank ρ belonging to S. The equations of the kinematic pairs determine the $6p$ Euler coordinates $\tilde{q}_1, \dots, \tilde{q}_{6p}$ of the system S as functions of

$$N = \sum_{\rho} \rho p_\rho \tag{5}$$

Lagrangian coordinates $q_1, \dots, q_N$. These latter are connected by transformation equations (in accordance with the results of (1), the transformation equations can be represented as equations involving only $q_1, \dots, q_N$). For each of the c independent simple closed contours of the system S, we have 12 transformation equations. Thus $q_1, \dots, q_N$ are subject to

$$K = 12(p-n) \tag{6}$$

equations

$$\alpha^{(0_\gamma, n_\gamma+1)}_{i_{0_\gamma} i_{n_\gamma+1}} = \sum_{i_{1_\gamma},\dots,i_{n_\gamma}} \prod_{\nu_\gamma=1}^{n_\gamma+1} \alpha^{(\nu_\gamma-1,\nu_\gamma)}_{i_{\nu_\gamma-1} i_{\nu_\gamma}}, \tag{7'}$$

$$x_{0_\gamma i_{0_\gamma}}^{K_{n_\gamma+1}} = \sum_{\nu_\gamma=1}^{n_\gamma+1} \left\{ \sum_{i_{1_\gamma},\ldots,i_{\nu_\gamma-1}} \left(\prod_{\nu'_\gamma=0}^{\nu_\gamma-1} \alpha_{i_{\nu'_\gamma-1} i_{\nu'_\gamma}}^{(\nu'_\gamma-1,\nu'_\gamma)} \right) x_{\nu_\gamma-1,i_{\nu_\gamma-1}}^{K_{\nu_\gamma}} \right\}, \qquad (7")$$

$$i_{0_\gamma}, \ldots = 1, 2, 3; \qquad \gamma = 1, \ldots, c$$

(the notation of (1) has been complicated by the introduction of the index γ, indicating the number of the contour; $1+n_\gamma$ is the number of links of the contour C_γ). In this case the number of independent equations of the system (7) cannot exceed $6c$.

The following considerations lead to a more precise result: composing, for each of the $c_{\rho'}$ simple closed contours of rank ρ', only ρ' independent transformation equations, we arrive at a system containing only some

$$k = \sum_{\rho'} \rho' c_{\rho'} \qquad (8)$$

of the equations of the system (7). However, the number of independent equations of the system, being equal to the rank R of the system, is, generally speaking, less than k. Thus the rank R of the system of transformation equations with respect to $q_1, \ldots, q_N$ satisfies the relation

$$R \leq \sum_{\rho'} \rho' c_{\rho'}. \qquad (9)$$

From the results established in (1), it follows that the rank R of the system of transformation equations with respect to the variables $q_1, \ldots, q_N$ does not depend on the choice of coordinate systems associated with $S_0, \ldots, S_n$. But it also does not depend on the choice of the system of independent simple closed contours constituting S. Thus, the rank of the system of transformation equations characte-

characterizes a property of the system S, immanent in this very system. This result makes it possible to construct a classification of the kinematic chains under consideration that reflects their actual geometric nature.

We shall call the rank R of the system of transformation equations the **rank of the system** S. Thus, we shall distinguish the kinematic chains under consideration according to their ranks. The classification system constructed in this way distinguishes kinematic chains by an essential feature and is, consequently, a natural system. The logical completeness of the system in question is determined by the fact that with the structural characteristic that has been found there is associated in this system an effective and, at the same time, quite general method of investigation.

The equations of system (7) determine some R of the variables $q_1, \dots, q_N$ as functions of the remaining $N-R$, whose values may be chosen arbitrarily. Accordingly, the decrement $d = N - R$ of the functional matrix of system (7) with respect to the variables $q_1, \dots, q_N$ indicates the number of degrees of freedom of the system $S_0 \dots S_n$. In particular, we shall have a system with one degree of freedom when $N = R + 1$.

Thus, equations (7) (together with the equations of the kinematic pairs) give a complete solution of the problem of the motion of a complex kinematic chain. Let us emphasize that the transformation equations, considered in their general form together with the equations of the kinematic pairs, contain all, without exception, characteristics of the motion of the system. The transformation equations are, therefore, the fundamental equations of the geometry of mechanisms. The results obtained create, however, the necessary prerequisites for applying the equations of analytical dynamics to the systems under consideration.

From the preceding it is clear that the number of degrees of freedom w of the system S is determined by the equation

$$w = \sum_{\rho} \rho p_{\rho} - R. \tag{10}$$

This is the most general structural equation for systems with integrable connections.

From (10), taking (9) into account, we shall have

$$w \geqslant \sum_{\rho} \rho p_{\rho} - \sum_{\rho'} \rho' c_{\rho'}. \tag{11}$$

Relation (11) is just as general as (10).

If the system S is formed by contours of one and the same rank r', then (11), in view of (4) and

$$p = \sum_{\rho} p_{\rho}, \tag{12}$$

passes into

$$w \geqslant r'n - \sum_{\rho} (r' - \rho) p_{\rho}. \tag{13}$$

This is a generalization of the well-known structural equation. From the preceding it is clear, however, that in the general case of systems formed by contours of different ranks, relation (13) is inapplicable.

Suppose that the system $S' \equiv S_1 \dots S_n$ is a group. Let $c^{(i)}$ be the number of independent simple closed contours and $p^{(e)}$ the number of external pairs of the system S'. If S' is attached to S_0 by all $p^{(e)}$ external pairs, then the number of independent simple closed contours of the system $S \equiv S_0 S_1 \dots S_n$ will be

$$c = c^{(i)} + p^{(e)} - 1. \tag{14}$$

Since, by the definition of a group, the number of degrees of freedom of the system S is equal to zero, (10) becomes

$$0 = \sum_{\rho} \rho p_{\rho} - R, \tag{15}$$

and (11) becomes

$$0 \geq \sum_{\rho} \rho p_{\rho} - \sum_{\rho'} \rho' c_{\rho'}. \tag{16}$$

Relation (13) becomes

$$0 \geq r' n - \sum_{\rho} (r' - \rho) p_{\rho}. \tag{17}$$

However, for groups formed by contours of one and the same rank r', the structural relation (16), in view of (14), may also be represented in the form

$$0 \geq \sum_{\rho} \rho p_{\rho} - r'(c^{(i)} + p^{(e)} - 1). \tag{18}$$

It should be noted that the characteristic $p^{(e)}$, which is customarily called the "order" of the group, enters into relations (14) and (18) and, consequently, plays a definite role in the structural investigation of groups. However, the so-called "class" of a group does not enter at all into the structural relations given above. From this point of view, the introduction of the concepts of the "class of a group" and the "class of a mechanism" appears unjustified.

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