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1958

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Abstract

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MATHEMATICS

A. M. VASIL' EV

ON ORTHOGONAL SUBGROUPS OF CLASSICAL COMPACT LIE GROUPS

(Presented by Academician P. S. Aleksandrov, 25 II 1958)

1. É. Cartan introduced into the space of a Lie group a quadratic differential form invariant with respect to left and right translations. For semisimple compact groups this form is positive definite. In the present note we set forth conditions for orthogonality, in the sense of the Cartan metric, of certain subgroups of a simple compact Lie group G. Knowledge of these conditions is important for certain problems in the geometry of homogeneous spaces, in particular for determining totally geodesic submanifolds of a homogeneous space in the sense of the Riemannian geometry induced on it by the Cartan metric of its group of motions G (cf. ([1])).

2. Let us specify the basic concepts. Two linear subspaces E, E' of a Euclidean N-dimensional vector space $E(N)$ intersect orthogonally if the orthogonal complement of their intersection $E \cap E'$ in E is completely orthogonal to E', and, consequently, the orthogonal complement of $E \cap E'$ in E' is completely orthogonal to E. In particular, if $E \subset E'$, we shall also regard them as intersecting orthogonally. Two submanifolds of a Riemannian space intersect orthogonally at a point M if their tangent spaces at M are orthogonal. Orthogonality of subgroups G_1, G_2 of a group G is understood in this sense at their intersection at the identity of the group.

3. The following proposition follows from Cartan' s theory of symmetric spaces ([2]). Let G_1 consist of all elements of G invariant with respect to an involutive automorphism I of it (I^2 is the identity automorphism). G_2 is orthogonal to G_1 if and only if some involutive automorphism of the group G_2 extends to an automorphism I of the group G. Here, among the automorphisms of G_2 one must include the identity automorphism and the mapping $g \to g^{-1}$.

4. In what follows, $O(n)$, $U(n)$, and $Sp(2n)$ denote, respectively, the group of all real orthogonal matrices of order n, the group of all complex unitary matrices of order n, and the group of all complex unitary-symplectic matrices of order $2n$. By $\widetilde{O}(n)$, $\widetilde{U}(2n)$, $\widetilde{Sp}(4n)$ we denote the simplest real representations of these groups in the spaces $E(n)$, $\widetilde{E}(2n)$, $\widetilde{E}(4n)$, respectively. We shall call them $\widetilde{R}$-representations. It is known ([3]) that $\widetilde{U}(2n)$

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may be regarded as the set of all orthogonal transformations of $\overline{E}(2n)$ preserving a certain decomposition of $\overline{E}(2n)$ into two-dimensional subspaces (a C-structure in $\overline{E}(2n)$). Similarly, $Sp(4n)$ for $n > 1$ may be specified by a certain decomposition of $\widetilde{E}(4n)$ into four-dimensional subspaces (a q-structure). $\widetilde{Sp}(4)$ in $\widetilde{E}(4)$ may be regarded as the set of orthogonal transformations inducing, in the improper hyperplane, the group of left (or right) Clifford translations ([4]). In this sense one may speak of opposite (left–right) q-structures in $\widetilde{E}(4)$.

A subspace E of the space $\bar{E}(2n)$ shall be called a C-subspace if every plane of the C-structure having a common ray with it belongs to it entirely. Thus a C-structure is also defined in E. In an analogous way one defines q-subspaces in $\widetilde{E}(4n)$, on each of which a definite q-structure is induced. A subspace E' in $\bar{E}(2n)$ (in $\widetilde{E}(4n)$) shall be called an O-subspace if every $E(2)$ of the C-structure (every $E(4)$ of the q-structure) has with it either a 0-dimensional or a one-dimensional intersection, and in the latter case this intersection is orthogonal to E'. A subspace $\bar{E}$ in $\widetilde{E}(4n)$ shall be called a C-subspace if every $E(4)$ of the q-structure has with it either a 0-dimensional or a 2-dimensional intersection, and in the latter case $E(4)$ is orthogonal to $\bar{E}$. These two-dimensional intersections define a C-structure in $\bar{E}$.

5. By a subgroup of type ρ of the group $\widetilde{O}(N)$ we shall mean a subgroup G_1 –a direct product of groups $O(l_a)$, $U(m_k)$, $Sp(2n_r)$, acting independently of one another [as $\widetilde{R}$-representations in a system of mutually completely orthogonal subspaces $E(l_a)$, $\bar{E}(2m_k)$, $\widetilde{E}(4n_r)$ (briefly, $E_a, \bar{E}_k, \widetilde{E}_r$) of the space $E(N)$ and leaving fixed all points of a maximal subspace $E(N_0)$, completely orthogonal to all $E_a, \bar{E}_k, \widetilde{E}_r$. Let a subgroup G_2 of type ρ in the same sense be associated with a system of subspaces $'E(l_{a'})$, $'\bar{E}(2m_{k'})$, $'\widetilde{E}(4n_{r'})$, $'E(N_0')$ (briefly, $'E_{a'}, '\bar{E}_{k'}, '\widetilde{E}_{r'}, 'E(N_0')$).

The following theorem is proved with the aid of the theory of linear representations of groups and associative algebras.

Theorem. The intersection $G_1 \cap G_2$ of two subgroups of type ρ of the group $\widetilde{O}(N)$ is a direct product of a certain number of groups $O(l_\alpha)$, $U(m_\varkappa)$, $Sp(2n_\rho)$, acting independently of one another in a completely orthogonal system of subspaces $E(l_\alpha\lambda_\alpha)$, $\bar{E}(2m_\varkappa\mu_\varkappa)$, $\widetilde{E}(4n_\rho\nu_\rho)$ (briefly, $E_\alpha, \bar{E}_\varkappa, \widetilde{E}_\rho$) as, respectively, λ_α-, $\mu_\varkappa$-, ν_ρ-fold $\widetilde{R}$-representations of these groups and leaving fixed all points of a maximal subspace $E(N_{00})$ (briefly, E_0), completely orthogonal to all $E_\alpha, \bar{E}_\varkappa, \widetilde{E}_\rho$. The commutator $[G_1 \cap G_2]$ (the totality of orthogonal transformations of $E(N)$ commuting with every transformation from $G_1 \cap G_2$) is a direct product of groups $O(\lambda_\alpha)$, $U(\mu_\varkappa)$, $Sp(2\nu_\rho)$, $O(N_{00})$, acting independently of one another in the same system of completely orthogonal subspaces as, respectively, l_α-, $m_\varkappa$-, n_ρ-, 1-fold $\widetilde{R}$-representations.

In $E(N)$ there exist two orthonormal bases B_1 and B_2 such that: a) every vector

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of B_1 belongs to the intersection of one of the subspaces $E_a, \bar{E}_k, \widetilde{E}_r, E(N_0)$ with one of the subspaces $E_\alpha, \bar{E}_\varkappa, \widetilde{E}_\rho, E_0$; b) B_2 is associated in an analogous way with the subspaces invariant with respect to G_2; c) the passage from B_1 to B_2 is accomplished by means of a transformation from $[G_1 \cap G_2]$.

Each of the intersections $\bar{E}_k$ (or $'\bar{E}_{k'}$) with $E_\alpha, \bar{E}_\varkappa, \widetilde{E}_\rho, E_0$ is a C-subspace in $\bar{E}_k$ (or $'\bar{E}_{k'}$). The intersections $\widetilde{E}_r$ and $'\widetilde{E}_{r'}$ with the same spaces are q-subspaces in $\widetilde{E}_r$ or $'\widetilde{E}_{r'}$.

6. Below we give some known propositions of the geometry of a Euclidean vector space $E(N)$ (see, for example, ([4]), Ch. I). Let two subspaces E, E' be given. Let the angle between the lines $e_1 \subset E$, $e_1' \subset E'$ take, as a function of e_1, e_1', stationary values $\varphi_\xi \leq 90^\circ$. To each φ_ξ correspond stationary subspaces $E_\xi \subset E$, $E_\xi' \subset E'$, and the E_ξ form a complete system of mutually completely orthogonal subspaces in E, and the E_ξ'— in E'. If $\varphi_\xi \neq 90^\circ$, then a natural one-to-one isometric correspondence is established between the points of E_ξ and E_ξ'. Subspaces E and E' having only one stationary angle φ,

are called (for $\varphi \neq 90^\circ$) paratactic. Subspaces E, E' having no more than two stationary angles φ_1, φ_2, with $\varphi_2 = 90^\circ$, and whose stationary subspaces E_1, E_1' corresponding to φ_1 are of dimension at most one, will be called almost orthogonal.

We shall consider l-dimensional subspaces $E(l)$, $2l$-dimensional subspaces $\bar{E}(2l)$, each of which carries a definite C-structure, and $4l$-dimensional subspaces $\widetilde{E}(4l)$, each of which carries a definite q-structure. We shall say that a pair of subspaces from this set is in O-position if they have only two stationary angles $\varphi_1 \neq 90^\circ$ and $\varphi_2 = 90^\circ$, and the stationary subspaces corresponding to φ_1 are l-dimensional and are O-subspaces (see § 4) in the C- or q-structure of the subspaces forming the pair. For example, $E(l)$ and $\bar{E}(2l)$, $\bar{E}(2l)$ and $\widetilde{E}(4l)$, $\widetilde{E}(4l)$ and $'\widetilde{E}(4l)$, etc., may be in O-position. The paratacticity of $E(l)$ and $'E(l)$ will be regarded as a special case of O-position.

Consider subspaces $\bar{E}(2m)$, carrying C-structures, and $\widetilde{E}(4m)$, carrying q-structures. We shall say that two subspaces from this set are in C-position if they have either only one $\varphi_1 \neq 90^\circ$, or only two stationary angles $\varphi_1 \neq 90^\circ$, $\varphi_2 = 90^\circ$, and the stationary subspaces corresponding to φ_1 are $2m$-dimensional and are C-subspaces whose C-structures naturally correspond to one another.

Let $\widetilde{E}(4l)$ carry a q-structure and let $E(l)$ be its O-subspace. The symmetry in $\widetilde{E}(4l)$ with respect to $E(l)$ takes its q-structure into some other q-structure. We shall agree to say that two subspaces $\widetilde{E}(4l)$ and $'\widetilde{E}(4l)$, carrying q-structures, are in O'-position if: a) they are paratactic; b) in the natural correspondence between these subspaces, the q-structure of $'\widetilde{E}(4l)$ corresponds in $\widetilde{E}(4l)$ to the q-structure obtained from its original q-structure by means of the symmetry

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with respect to some l-dimensional O-subspace.

7. Below are listed all necessary and sufficient conditions for the orthogonality of two subgroups of type ρ.

A. The dimension of the intersection of the space E_α with the spaces E_a, $'E_{a'}$ must be either 0 or l_α, with $\bar{E}_k$, $'\bar{E}_{k'}$ either 0 or $2l_\alpha$, and with $\widetilde{E}_r$, $'\widetilde{E}_{r'}$ either 0 or $4l_\alpha$. We denote the nonzero intersections of E_χ with E_a by $E_{\alpha a}$, with $'E_{\alpha'}$ by $'E_{\alpha\alpha'}$, with $\bar{E}_k$ by $\bar{E}_{\alpha k}$, etc. Each of the spaces $E_{\alpha a}, \bar{E}_{\alpha k}, \widetilde{E}_{\alpha r}$ must be in O-position with each $'E_{\alpha a'}, 'E_{\alpha k'}$, and each $E_{\alpha a}, \bar{E}_{\alpha k}$ with each $'\widetilde{E}_{\alpha r'}$. The spaces $\widetilde{E}_{\alpha r}$ and $'\widetilde{E}_{\chi r'}$ are either in O-position or in O'-position (see § 6). For $\alpha =$ const, the stationary angle between any two $E_{\alpha a}, 'E_{\alpha a'}$ is one and the same and is equal to x_α; the stationary angle, distinct from 90°, between any two $E_{\alpha a}, '\bar{E}_{\alpha k'}$ or $'E_{\alpha a'}, \bar{E}_{\alpha k}$ is one and the same and is equal to u_χ; the same is true for any $E_{\alpha a}, '\widetilde{E}_{\alpha r'}$ and $'E_{\alpha a'}, \widetilde{E}_{\alpha r}$—their angle is v_α, for any $\bar{E}_{\alpha k}, '\widetilde{E}_{\alpha r'}$ and $'\bar{E}_{\alpha k'}, \widetilde{E}_{\alpha r}$—the angle is w_α, and for $\widetilde{E}_{\alpha r}$ and $'\widetilde{E}_{\alpha r'}$ their angle is, in the case of O-position, z_α, and in the case of O'-position, z'_α. The relations must hold

$$\cos z_\alpha = \sqrt{2}\cos w_\alpha = 2\cos z'_\alpha = 2\cos v_\alpha = 2\cos y_\alpha = 2\sqrt{2}\cos u_\alpha = 4\cos x_\alpha.$$

The space $\bar{E}_\chi$ may have nonzero intersections only with $\bar{E}_k, '\bar{E}_{k'}$—of dimensions $2m_\chi$, and with $\widetilde{E}_r, '\widetilde{E}_{r'}$—of dimensions $4m_\chi$. Denote these intersections by $\bar{E}_{\chi k}, '\bar{E}_{\chi k'}, \widetilde{E}_{\chi r}, '\widetilde{E}_{\chi r'}$. Each $\bar{E}_{\chi k}, \widetilde{E}_{\chi r}$ must be in C-position (see § 6) with each $'\bar{E}_{\chi k'}, '\widetilde{E}_{\chi r'}$. For $\chi =$ const, the stationary angle between $\bar{E}_{\chi k}, '\bar{E}_{\chi k'}$ is one and the same and is equal to y_χ; distinct from 90°

stationary angle between $\bar{E}_{\chi k}$, $'\bar{E}_{\chi r'}$ or $'\bar{E}_{\chi k'}$, $\hat{E}_{\chi r}$ is one and the same, equal to ω_χ; the same is true for $\tilde{E}_{\chi r}$, $'\bar{E}_{\chi r'}$—their angle is z_χ. The relations $\cos z_\chi = \sqrt{2}\cos\omega_\chi = 2\cos y_\chi$ must be satisfied. Each $\tilde{E}_\rho$ can have nonzero intersections only with $\tilde{E}_r$, $'\tilde{E}_{r'}$ of dimensions $4n$ (denote them by $\hat{E}_{\rho r}$, $'\bar{E}_{\rho r'}$). For $\rho =$ const each pair $\hat{E}_{\rho r}$, $'\bar{E}_{\rho r'}$ is paratactic, their stationary angles are identical, and the q-structures of these subspaces, by virtue of paratacticity, correspond to one another.

B. No pair of subspaces, one of which belongs to the series E_a, $\bar{E}_k$, $\tilde{E}_r$, and the other to the series $'E_{a'}$, $'\bar{E}_{k'}$, $'\tilde{E}_{r'}$, can simultaneously have nonzero intersections with two subspaces from the series E_α, $\bar{E}_\chi$, $\tilde{E}_\rho$. If the indicated pair of subspaces has nonzero intersections with one of the subspaces of the series E_α, $\bar{E}_\chi$, $\tilde{E}_\rho$, it must intersect E_0 in a pair of subspaces completely orthogonal to each other.

C. Denote by E_{0a}, $'E_{0a'}, \ldots$, $'\tilde{E}_{0r'}$ the intersections of the subspaces E_a, $'E_{a'}, \ldots$, $'E_{r'}$ with E_0. Each subspace E_{0a}, $\bar{E}_{0k}$, $\tilde{E}_{0r}$ must be almost orthogonal to each $'E_{0a'}$, $'\bar{E}_{0k'}$, and each E_{0a}, $\bar{E}_{0k}$—to each $'\tilde{E}_{0r'}$. The subspaces $\tilde{E}_{0r}$, $'\tilde{E}_{0r'}$ must either be almost orthogonal, or have two stationary

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angles $\varphi_1 \neq 90^\circ,\ \varphi_2 = 90^\circ$, with the stationary subspaces corresponding to φ_1 being 4-dimensional q-subspaces in $\tilde{E}_{0r},\ {}'\tilde{E}_{0r'}$. By virtue of the correspondence between these paratactic subspaces, their q-structures must be opposite (see item 4).

8. All configurations of the subspaces and of their C- and q-structures that occur in item 8 are in fact realized for arbitrary $l_a,\ m_\chi,\ n_\rho,\ N_{00}$ and for $\lambda_\alpha,\ \mu_\chi,\ \nu_\rho$ equal to powers of the number 2 (including, in some cases, the zeroth power).

9. If two subgroups of type ρ belong to some subgroup of the group $\tilde{O}(N)$, for example $\tilde{U}(N)$ or $\tilde{S}p(N)$, the formulation of the conditions for their orthogonality is greatly simplified.

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Received
11 II 1958

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Note: Figure translations are in progress. See original paper for figures.

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