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MATHEMATICS

1958

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Abstract

Full Text

MATHEMATICS

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ON THE DECOMPOSITION OF IRREDUCIBLE REPRESENTATIONS OF THE PRINCIPAL SERIES OF THE COMPLEX UNIMODULAR GROUP OF ORDER n INTO REPRESENTATIONS OF THE COMPLEX UNIMODULAR GROUP OF SECOND ORDER

(Presented by Academician S. L. Sobolev, 2 IV 1958)

Let $g \rightarrow T_g$ be a continuous irreducible unitary representation of a topological group G , and let H be a certain subgroup of the group G . The restriction of the representation $g \rightarrow T_g$ to the group H is a unitary representation $h \rightarrow T_h$ of the group H , generally speaking reducible. The problem arises of decomposing the representation $h \rightarrow T_h$ into irreducible representations of the group H .

The results of the author's preceding paper ⁽¹⁾ make it possible to solve this problem for the case when G is the complex unimodular group A_n of order n , $g \rightarrow T_g$ is a representation of the principal series of this group, and H is the group of all matrices $g \in A_n$ such that $g_{pq} = \delta_{pq}$ for $p > 2$ or $q > 2$ (δ_{pq} is the Kronecker symbol), so that H is isomorphic to the group A_2 .

For simplicity of exposition we shall confine ourselves here to the case $n = 3$. In this case (see ⁽²⁾, item 2, § 5) the representations of the principal series are realized in the Hilbert space $L^2(Z_3)$ of all measurable functions $f(z) = f(z_{21}, z_{31}, z_{32})$ on the group Z_3 —the group of all matrices

$$z = \begin{vmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{vmatrix}, \tag{1}$$

satisfying the condition*

$$\|f\|^2 = \int |f(z)|^2 dz = \int |f(z_{21}, z_{31}, z_{32})|^2 dz_{21} dz_{32} dz_{32} < \infty,$$

and the operators T_g of the representation are given by the formula

$$T_g f(z) = |g_2^1|^{m_1+i\rho_1-2} (\overline{g_2^1})^{-m_1} |g_3^1|^{m_2-m_1+i(\rho_2-\rho_1)-2} (\overline{g_3^1})^{-m_2+m_1} f(z\bar{g}), \quad (2)$$

where g_p^1 is the minor formed from the last p rows and columns of the matrix $g^1 = zg$; $z\bar{g}$ is the matrix $z^1 \in Z_3$, determined by the relation $zg = kz^1$; $k_{pq} = 0$ for $p > q$; m_1, m_2 are integers; ρ_1, ρ_2 are real numbers determining the given representation $g \rightarrow T_g$ (the parameters of the representation).

* As in the preceding paper ⁽¹⁾, $\int \varphi(z_1, z_2, \dots, z_m) dz_1 \cdots dz_m$, where z_1, \dots, z_m are complex variables, denotes the $2m$ -fold integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1 + iy_1, \dots, x_m + iy_m) dx_1 dy_1 \cdots dx_m dy_m.$$

If, in particular, $g \in H$, i.e.

$$g = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

then, by virtue of (1),

$$g^1 = zg = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{11}z_{21} + g_{21} & g_{12}z_{21} + g_{22} & 0 \\ g_{11}z_{31} + g_{21}z_{32} & g_{12}z_{31} + g_{22}z_{32} & 1 \end{vmatrix}, \quad (3)$$

hence $g_2^1 = g_{12}z_{21} + g_{22}$, $g_3^1 = 1$. Moreover, applying to (3) formula (5.22) §5 in ⁽²⁾, we obtain that

$$z_{21}^1 = \frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \quad z_{31}^1 = g_{11}z_{31} + g_{21}z_{32}, \quad z_{32}^1 = g_{12}z_{31} + g_{22}z_{32},$$

and substitution in (2) shows that the restriction of the given representation to the group H is given by the formula

$$\begin{aligned} T_g f(z_{21}, z_{31}, z_{32}) &= \\ &= |g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} \times \\ &\times f \left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, g_{11}z_{31} + g_{21}z_{32}, g_{12}z_{31} + g_{22}z_{32} \right). \end{aligned} \quad (4)$$

Formula (4) means that the representation $g \rightarrow T_g$ of the group H is the tensor product of the irreducible representation $g \rightarrow T_g^{(1)}$ of the group A_2 , where

$$T_g^{(1)} f_1(z) = |g_{12}z + g_{22}|^{m_1+i\rho_1-2} (g_{12}z + g_{22})^{-m_1} f_1\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right),$$

and the quasiregular representation (see (3), p. 421) $g \rightarrow T_g^{(2)}$ of this group:

$$T_g^{(2)} f_2(z_1, z_2) = f_2(g_{11}z_1 + g_{21}z_2, g_{12}z_1 + g_{22}z_2). \quad (5)$$

Therefore we shall obtain the decomposition of the representation (4) into irreducible representations if we first decompose into irreducible representations $g \rightarrow T_g^X$ the quasiregular representation (5), and then decompose into irreducible representations the tensor products $T_g^{(1)} \times T_g^X$.

To obtain the first decomposition, put

$$\zeta = \frac{z_{31}}{z_{32}}, \quad f(z_{21}, z_{31}, z_{32}) = f(z_{21}, \zeta, z_{32}) = \varphi(z_{21}, \zeta, z_{32}) |z_{32}|^{-2}, \quad (6)$$

$$\psi(z_{21}, \zeta, \chi') = \int \varphi(z_{21}, \zeta, z_{32}) |z_{32}|^{-2} \chi'(z_{32}) dz_{32}, \quad (7)$$

where $\chi'(z) = |z|^{m'+i\rho'} z^{-m'}$. The correspondence $f \rightarrow \psi$, established by formulas (6), (7), is an isometric mapping U of the space $L^1(Z_3)$ onto the Hilbert space \mathfrak{H}_ψ of all measurable functions $\psi(z_{21}, \zeta, \chi')$ satisfying the condition

$$\|\psi\|^2 = \int |\psi(z_{21}, \zeta, \chi')|^2 dz_{21} d\zeta d\chi' < \infty;$$

here the integral with respect to $d\chi'$ is taken over the whole group of characters X of the multiplicative group of complex numbers, and $d\chi'$ denotes $\frac{1}{(2\pi)^2} d\rho'$. This mapping U transforms the operators T_g into operators in the space \mathfrak{H}_ψ , which we shall again denote by T_g .

But, under the passage from f to $T_g f$ according to formula (4), the function $\varphi(z_{21}, \zeta, z_{32})$ passes into

$$\begin{aligned} & |z_{32}|^2 |g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} + \\ & + \varphi\left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \frac{g_{11}z_{31} + g_{21}z_{32}}{g_{12}z_{31} + g_{22}z_{32}}, g_{12}z_{31} + g_{22}z_{32}\right) |g_{12}z_{31} + g_{22}z_{32}|^{-2} = \end{aligned}$$

$$= |g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} |g_{21}\zeta + g_{22}|^{-2} \times \\ \times \varphi \left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \frac{g_{11}\zeta + g_{21}}{g_{12}\zeta + g_{22}}, (g_{12}\zeta + g_{22})z_{32} \right),$$

therefore, $\psi(z_{21}, \zeta, \chi')$ passes into

$$|g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} |g_{12}\zeta + g_{22}|^{-2} \times \\ \times \int \varphi \left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \frac{g_{11}\zeta + g_{21}}{g_{12}\zeta + g_{22}}, (g_{12}\zeta + g_{22})z_{32} \right) |z_{32}|^{-2} \overline{\chi'(z_{32})} dz_{32} = \\ = |g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} |g_{12}\zeta + g_{22}|^{-2} \times \\ \times \int \varphi \left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \frac{g_{11}\zeta + g_{21}}{g_{12}\zeta + g_{22}}, z_{32} \right) |z_{32}|^{-2} \overline{\chi'((g_{12}\zeta + g_{22})^{-1}z_{32})} dz_{32} = \\ = |g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} |g_{12}\zeta + g_{22}|^{m'+i\rho'-2} (g_{12}\zeta + g_{22})^{-m'} \times \\ \times \psi \left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \frac{g_{11}\zeta + g_{21}}{g_{12}\zeta + g_{22}}, \chi' \right),$$

so that

$$T_g \psi(z_{21}, \zeta, \chi') = |g_{12}z_{21} + g_{22}|^{m_1+i\rho_1-2} (g_{12}z_{21} + g_{22})^{-m_1} |g_{12}\zeta + g_{22}|^{m'+i\rho'-2} \times \\ \times (g_{12}\zeta + g_{22})^{-m'} \psi \left(\frac{g_{11}z_{21} + g_{21}}{g_{12}z_{21} + g_{22}}, \frac{g_{11}\zeta + g_{21}}{g_{12}\zeta + g_{22}}, \chi' \right). \quad (8)$$

Formula (8) means that the representation $g \rightarrow T_g$ in \mathfrak{H}_ψ is the continuous sum over χ' of tensor products $T_g^{(1)} \times T_g^{\chi'}$, where

$$T_g^{\chi'} f(\zeta) = |g_{12}\zeta + g_{22}|^{m'+i\rho'-2} (g_{12}\zeta + g_{22})^{-m'} f \left(\frac{g_{11}\zeta + g_{21}}{g_{12}\zeta + g_{22}} \right).$$

Therefore it remains to carry out the decomposition into irreducible representations of each of these tensor products according to the formulas of the article [1]. For this purpose denote by \mathfrak{F} the set of all pairs $(\chi, \chi') \in X \times X$ for which $m_1 + m' + m$ is an even integer, and put, for $(\chi, \chi') \in \mathfrak{F}$,

$$\begin{aligned}
 F(z, \chi, \chi') = F(z, m, \rho, m', \rho') &= \int \psi(z_{21}, \zeta, \chi') |\zeta - z_{21}|^{-\frac{m+m_1+m'}{2} - i\frac{\rho+\rho_1+\rho'}{2}} \times \\
 &\times (\zeta - z_{21})^{\frac{m+m_1+m'}{2}} |z - z_{21}|^{\frac{m-m_1+m'}{2} + i\frac{\rho-\rho_1+\rho'}{2} - 1} (z - z_{21})^{-\frac{m-m_1+m'}{2}} \times \\
 &\times |\zeta - z|^{\frac{m+m_1-m'}{2} + i\frac{\rho+\rho_1-\rho'}{2} - 1} (\zeta - z)^{-\frac{m+m_1+m'}{2}} dz_{21} d\zeta; \quad (9)
 \end{aligned}$$

the correspondence $\psi \rightarrow F$, established by formula (9), is an isometric mapping of the space \mathfrak{H}_ψ onto the Hilbert space \mathfrak{H}_F of all measurable functions $F(z, \chi, \chi')$, $(\chi, \chi') \in \mathfrak{F}$, satisfying the conditions:

1)

$$\|F\|^2 = \frac{1}{4\pi^2} \iint_{\mathfrak{F}} (m^2 + \rho^2) \left[\int |F(z, \chi, \chi')|^2 dz \right] d\chi d\chi' < \infty;$$

2) the Fourier transforms

$$\tilde{F}(w, \chi, \chi') = (2\pi)^{-2} \int F(z, \chi, \chi') \exp(-i \operatorname{Re}(z\bar{w})) dz$$

of the functions $F(z, \chi, \chi')$ for almost all w, χ, χ' , $(\chi, \chi') \in \mathfrak{F}$, satisfy relation

$$\begin{aligned}
 \tilde{F}(w, \chi^{-1}, \chi') &= (-i)^{-m_2 - i\rho} \times \\
 &\times \frac{\Gamma\left(\frac{m+m_1-m'}{2} + \frac{1}{2} - i\frac{\sigma+\sigma_1-\sigma'}{2}\right) \Gamma\left(\frac{-m-m_1+m'}{2} + \frac{1}{2} - i\frac{\sigma-\sigma_1+\sigma'}{2}\right)}{\Gamma\left(\frac{m+m_1+m'}{2} + \frac{1}{2} + i\frac{\sigma+\sigma_1-\sigma'}{2}\right) \Gamma\left(\frac{-m-m_1+m'}{2} + \frac{1}{2} + i\frac{\sigma-\sigma_1+\sigma'}{2}\right)} \times \\
 &\times |w|^{-m+i\rho} w^m \tilde{F}(w, \chi, \chi').
 \end{aligned}$$

Under this mapping the representation $g \rightarrow T_g$ goes over into a representation in \mathfrak{H}_F , which we shall again denote by $g \rightarrow T_g$; from the preceding arguments and the results of paper ¹ it follows that the representation $g \rightarrow T_g$ in the space \mathfrak{H}_F is given by the formula

$$T_{gF}(z, \chi, \chi') = |g_{12}z + g_{22}|^{m+i\rho-2} (g_{12}z + g_{22})^{-m} F\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}, \chi, \chi'\right),$$

i.e. this representation is a continuous sum of representations $T_g^{(\chi)}$. Consequently, combining formulas (6), (7), and (9), we arrive at the following result:

Theorem. For every function $f \in L^2(Z_3)$ the integral

$$\begin{aligned} F(z, \chi, \chi') &= \int f(z_{21}, z_{31}, z_{32}) |z_{31} - z_{32}z_{21}|^{-\frac{m+m_1+m'}{2} - i\frac{\rho+\rho_1+\rho'}{2} - 1} \times \\ &\times (z_{31} - z_{32}z_{21})^{\frac{m+m_1+m'}{2}} |z - z_{21}|^{\frac{m-m_1+m'}{2} + i\frac{\rho-\rho_1+\rho'}{2} - 1} (z - z_{21})^{-\frac{m-m_1+m'}{2}} \times \\ &\times |z_{21} - z_{32}z|^{-\frac{m+m_1-m'}{2} + i\frac{\rho+\rho_1-\rho'}{2} - 1} (z_{21} - z_{32}z)^{-\frac{m+m_1-m'}{2}} dz_{21} dz_{31} dz_{32} \end{aligned} \quad (10)$$

converges in the sense of the norm in \mathfrak{H}_F , and formula (10) realizes an isometric mapping of the space $L^2(Z_3)$ onto the space \mathfrak{H}_F . Under this mapping, the restriction to the group $H \sim A_2$ of the representation $g \rightarrow T_g$ of the principal series of the group A_3 , defined by formula (2), goes over into a continuous sum of irreducible representations of the principal series $g \rightarrow T_g^{(\chi)}$ of the group A_2 , so that in passing from f to T_{gf} the function $F(z, \chi, \chi')$ goes over into

$$T_g^{(\chi)} F(z, \chi, \chi') = |g_{12}z + g_{22}|^{m+i\rho-2} (g_{12}z + g_{22})^{-m} F\left(\frac{g_{11}z + g_{12}}{g_{21}z + g_{22}}, \chi, \chi'\right),$$

where $\chi(\lambda) = |\lambda|^{m+i\rho} \lambda^{-m}$.

We note that the theorem proved is applicable to other subgroups of the group A_3 that are isomorphic to the group A_2 . For example, in the case of the subgroup of matrices

$$g = \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{array} \right\|$$

the corresponding decomposition is obtained from the decomposition for the subgroup H and the relation $sH's = H$, where

$$s = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

Received
31 III 1958

CITED LITERATURE

¹ M. A. Naimark, DAN, **119**, No. 5 (1958). ² I. M. Gel' fand, M. A. Naimark, Tr. Matem. inst. im. V. A. Steklova, **36**, 1 (1956). ³ I. M. Gel' fand, M. A. Naimark, Izv. AN SSSR, ser. matem., **11**, 411 (1947).

Note: Figure translations are in progress. See original paper for figures.

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