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Abstract

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SOME PROBLEMS ON LARGE DEVIATIONS OF THE MAXIMUM OF SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables. Denote:

$$\mathbf{M}\xi_k = m, \qquad s_n = \sum_{k=1}^{n} \xi_k, \qquad \bar{s}_n = \max_{0 \le \nu \le n} s_\nu.$$

It is required to study the asymptotic behavior of the probability $\mathbf{P}(\bar{s}_n < x)$ as $n \to \infty$, where x may depend on n (when it is necessary to emphasize this dependence, we shall write $x(n)$). It will be more convenient to formulate the problem as one concerning the time of first passage through the barrier at the point 0 in a homogeneous random walk, when the wandering particle starts from the point $x > 0$. A complete solution of this question, when the jump size ξ_k can take only the two values ± 1 with probabilities p and $1-p$, is contained, for example, in Feller' s book ([1]). The method used there for finding the explicit form of the generating function of the time of first passage does not extend to a more general type of distribution of the jumps. Therefore the behavior of the generating function near its first singular points was studied; with the use of the saddle-point method ([2], [3]), this gives the required asymptotic expansion for the probabilities. Below we give the results for the case in which the ξ_k are bounded and have a lattice distribution.

We shall assume that ξ_k can take only the integer values $l_1, l_2, \ldots, l_q$ ($l_1 = -r < 0;\ l_{i+1} > l_i,\ i = 1, 2, \ldots, q-1;\ l_q = s > 0$) with probabilities $p_{l_i} = \mathbf{P}(\xi_k = l_i)$, and that the greatest common divisor of the numbers $l_1, l_2, \ldots, l_q$ is equal to 1. Let $\Delta_i = l_{i+1} - l_i,\ i = 1, 2, \ldots, q-1;\ d = (\Delta_1, \Delta_2, \ldots, \Delta_{q-1})$, and let l be the residue of any of the numbers l_i modulo d ($(l, d) = 1;\ 0 \le l \le d-1$). We shall further denote by $w(\lambda)$ the "generating" function of the jump

$$w(\lambda) = \sum_{i=-r}^{s} p_i \lambda^i$$

and by λ_1 the point at which $w(\lambda)$ attains its minimum on the ray $\lambda > 0$.

Consider the equation $w(\lambda) - w(\lambda_1) = 0$. Using Pellet' s theorem ([4]), one can show that this equation contains in the circle $|\lambda| < \lambda_1$ exactly $r-1$ roots $\lambda_2, \lambda_3, \dots, \lambda_r$, counted with their multiplicities. For simplicity, we shall assume that among the roots $\lambda_2, \lambda_3, \dots, \lambda_r$ there are no multiple roots. Then one may put

$$\sigma_i = \sum_{\substack{k=1\\k\neq i}}^{r} \frac{1}{\lambda_i - \lambda_k}, \qquad \pi_i = \prod_{\substack{k=1\\k\neq i}}^{r} (\lambda_i - \lambda_k), \qquad \varepsilon_i = \prod_{\substack{k=1\\k\neq i}}^{r} (1 - \lambda_k).$$

Moreover, let $f_{ij}(p)$, $f_i(p)$ be the elementary symmetric polynomials of order p in all the roots $\lambda_1, \lambda_2, \dots, \lambda_r$, except respectively λ_i, λ_j.

and λ_i. We shall assume that $f_{ij}(p) = 1$, $f_i(p) = 1$, if $p = 0$, and $f_{ij}(p) = 0$, if $p < 0$. Then, for integral x, the following theorem is true.

Theorem. If $u_{x,n}$ is the probability of absorption at the n-th step by the screen at the point 0 of a particle starting from the point $x > 0$, then:

A. For x independent of n,

$$u_{x,n} = \frac{[w(\lambda_1)]^{n+1}}{n^{3/2}} \frac{d}{\sqrt{2\pi w(\lambda_1) w''(\lambda_1)}} b_0(x,n) \left\{ 1 + \frac{b_1(x,n)}{n} + \frac{b_2(x,n)}{n^2} + \dots + O\left(\frac{1}{n^k}\right) \right\},$$

where $b_0(x,n)$ is a function periodic in n with period d, equal to

$$\begin{aligned} b_0(x,n) =& \frac{\lambda_1^{r+x-1}}{\pi_1} \left\{ \frac{r+x+1}{\lambda_1} - \sigma_1 \right\} \sum_{k=0}^{\left[\frac{r-1-c(n)}{d}\right]} (-1)^{c(n)+kd} f_1(c(n)+kd) \\ &+ \sum_{i=2}^{r} \frac{\lambda_i^{r+x-1}}{\pi_i} \sum_{k=0}^{\left[\frac{r-1-c(n)}{d}\right]} (-1)^{c(n)+kd} \left\{ f_{i1}(c(n)+kd-i) + \frac{f_i(c(n)+kd)}{\lambda_i - \lambda_1} \right\}. \end{aligned}$$

Here $c(n) \equiv ln - x \pmod{d}$, $0 \le c(n) \le d-1$, and $[y]$ denotes the integer part of y; $b_i(x,n)$ also have period d in n, are bounded, and can be calculated; k is any natural number.

B*. For $x(n) \to \infty$, $n \to \infty$ and $x(n) = o(\sqrt{n})$,

$$u_{x,n} = \frac{[w(\lambda_1)]^{n+1}}{n^{3/2}} x \lambda_1^{x+r-2} \frac{d}{\pi_1 \sqrt{2\pi w(\lambda_1) w''(\lambda_1)}} \left\{ \sum_{k=0}^{\left[\frac{r-1-c(n)}{d}\right]} (-1)^{c(n)+kd} f_1(c(n)+kd) \right\}$$

$$\times\left(1+O\left(\frac{1}{x}\right)+O\left(\frac{x^2}{n}\right)\right).$$

C. For $\dfrac{\sqrt{n}}{x(n)}=O(1)$ and $x(n)=o(n)$,

$$u_{x,n}=\frac{[w(\lambda_1)]^{n+1}}{n^{3/2}}\,x\lambda_1^{x+r-2}e^{nh(x/n)}\frac{d}{\pi_1\sqrt{2\pi w(\lambda_1)w''(\lambda_1)}}\times$$

$$\times\left\{\sum_{k=0}^{\left[\frac{r-1-c(n)}{d}\right]}(-1)^{c(n)+kd}f_1(c(n)+kd)\right\}\left(1+O\left(\frac{x}{n}\right)\right).$$

Here $h(t)$ is a power series convergent for sufficiently small values of $|t|$:

$$h(t)=-\frac{w(\lambda_1)}{w''(\lambda_1)}\frac{t^2}{2\lambda_1^2}-\left(\frac{w(\lambda_1)}{w''(\lambda_1)}\right)^2\left[\frac{w'''(\lambda_1)}{w''(\lambda_1)}+\frac{3}{\lambda_1}\right]\frac{2t^3}{3\lambda_1^3}+\cdots.$$

If among $\lambda_2,\dots,\lambda_r$ there are multiple roots, then only the expressions for $b_i(x,n)$ in item A change.

For $d=1$ the formulas simplify: $b_i(x,n)\equiv b_i(x)$, and $b_0(x)$ can be written in the form

$$b_0(x)=\lambda_1^{r+x-1}\frac{\varepsilon_1}{\pi_1}\left(\frac{r+x-1}{\lambda_1}-\sigma_1\right)+\sum_{i=2}^{r}\lambda_i^{r+x-1}\frac{\varepsilon_i}{\pi_i}\left(\frac{1}{\lambda_i-\lambda_1}-\frac{1}{1-\lambda_1}\right),$$

and the expression in braces in the formulas of items B and C is replaced by ε_1/π_1.

For $x(n)\sim cn$ with $c<r$ and $c=r$, analogous theorems can also be formulated.

* In items B, C we restrict ourselves only to the first terms of the asymptotic expansion. The notation $c(n)$ will have the same meaning as in item A.

Next, let us consider $p(x)$—the probability that the wandering point will at some time be absorbed by the screen at the point 0. $p(x)$ differs from 1 only when the mean jump is directed away from the absorbing screen. If we set

$$\Pi_i=\prod_{\substack{k=1\\k\neq i}}^{r}(\mu_i-\mu_k),\qquad E_i=\prod_{\substack{k=1\\k\neq i}}^{r}(1-\mu_k),$$

then

$$p(x) = \sum_{i=1}^{r} \mu_i^{x+r-1} \frac{E_i}{\Pi_i},$$

where $\mu_1, \mu_2, \dots, \mu_r$ are the roots of the equation $w(\mu) - 1 = 0$ such that $|\mu_i| < 1$ (in view of (2) there will be exactly r such roots; among them there is always a positive one, maximal in modulus). For simplicity we again assume that none of the roots $\mu_1, \mu_2, \dots, \mu_r$ is multiple.

Now put $\xi_k' = -\xi_k$ and denote by $u_{x,n}'$, $p'(x)$ the absorption probabilities corresponding to the random variable ξ_k'; then

$$u_{x,n}' = \mathbf{P}(\bar{s}_{n-1} < x, \bar{s}_n \geq x)$$

and

$$1 - p'(x) = \mathbf{P}(\bar{s}_i < x,\ i = 1, 2, \dots).$$

For $m \geq 0$, the desired probability $\mathbf{P}(\bar{s}_n < x(n))$ can be computed as

$$\sum_{k=n+1}^{\infty} u_{x(n),k}'.$$

For $m < 0$,

$$\mathbf{P}(\bar{s}_n < x(n)) = 1 - p'(x(n)) + \sum_{k=n+1}^{\infty} u_{x(n),k}'.$$

If a second absorbing screen is introduced at the point $a > 0$ and one again considers $u_{x,n}$—the probability of absorption at the n-th step by the screen at the point 0 of a particle starting from a point $0 < x < a$—then the asymptotic behavior of $u_{x,n}$ will be different. It turns out that in this case the generating function

$$U_x(z) = \sum_{n=0}^{\infty} u_{x,n} z^n$$

is a rational function $U_x(z) = P_x(z)/Q(z)$, analytic in a disk of radius $\rho > 1/w(\lambda_1)$. The polynomials $P_x(z)$ and $Q(z)$ can be found.

Some conclusions concerning the probability $u_{x,n}$ with one absorbing screen can also be drawn for other types of distribution functions of the random variables ξ_k. We shall assume that Cramér' s condition [(5)] is satisfied, i.e., there exists an interval $[\alpha, \beta]$, $\alpha < 1$, $\beta > 1$, on which $w(\lambda)$ exists, and, moreover, the point λ_1

at which $w(\lambda)$ attains its minimum is an interior point of $[\alpha, \beta]$. Then the use of the results of works [5–7] and Lemma 5 of work [8] permits one to assert that, for a sufficiently broad class of distribution functions of the jump and $x = o(n)$,

$$\lim_{n\to\infty} \frac{\log u_{x,n}}{n} = \log w(\lambda_1).$$

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Note: Figure translations are in progress. See original paper for figures.