MATHEMATICS

S. A. CHUNIKHIN

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. A. CHUNIKHIN

ON THE PERMUTABILITY OF FACTORS IN Π-FACTORIZATIONS OF FINITE GROUPS

(Presented by Academician I. M. Vinogradov, 7 XII 1957)

§ 1. In the papers (1,2) we introduced the so-called Π-factorization of finite groups, and it was shown that every finite group admits factorizations of this kind in a definite way. In the present paper it is established that among the Π-factorizations of finite groups there always exist Π-factorizations with pairwise permutable factors. Thus Theorem 2 of (1,2) receives its natural strengthening.

§ 2. In the present article we shall use the definitions and notation introduced in (1,2). In particular, $\mathfrak{G}$ denotes a certain finite group of order g. If n, n_1, and n/n_1 are natural numbers and $(n_1, n/n_1) = 1$, then n_1 will be called a Sylow divisor of the number n.

Lemma 1. Let h be a certain reduced Π-integral divisor of the order g of the group $\mathfrak{G}$, and let $\mathfrak{A}$ be a certain subgroup of order a of the group $\mathfrak{G}$. Then $b = (a, h)$ will be a certain reduced Π-integral divisor of a.

Lemma 2. If $m = h_1 h_2 \dots h_t$ is the greatest Π-divisor of the number g, and $h_1, h_2, \dots, h_t$ are reduced Π-integral divisors of g, then $h_1, h_2, \dots, h_t$ are pairwise relatively prime.

Lemma 3. Let m and $h_1, h_2, \dots, h_t$ be the same as in Lemma 2. If $\alpha_i = (a, h_i)$, $i = 1, 2, \dots, t$, then $\alpha = \alpha_1 \alpha_2 \dots \alpha_t$ will be the greatest Π-divisor of the order a of the subgroup $\mathfrak{A}$, and the numbers $\alpha_1, \alpha_2, \dots, \alpha_t$ will be pairwise relatively prime reduced Π-integral divisors of a.

Lemma 4. If subgroups $\mathfrak{A}$ and $\mathfrak{B}$ of the group $\mathfrak{G}$ are permutable and the greatest Π-divisors of their orders divide the Sylow divisor v of the number g, then the greatest Π-divisor of the order of the subgroup $\mathfrak{A}\mathfrak{B}$ also divides v.

Lemma 5. Let $\mathfrak{G} = \mathfrak{A}_1 \mathfrak{A}_2 \dots \mathfrak{A}_r$, where $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_r$ are pairwise permutable subgroups of respective orders $a_1, a_2, \dots, a_r$. Suppose that the greatest Π-divisor a'_ρ of the order of the subgroup $\mathfrak{A}_\rho$ divides the Π-divisor s of the number g, and moreover s is relatively prime to each of the numbers $a_1, a_2, \dots, a_{\rho-1}, a_{\rho+1}, \dots, a_r$. Then $a'_\rho = s$.

Lemma 6. Let a normal divisor $\mathfrak{S}$ of the group $\mathfrak{G}$ contain the subgroup $\mathfrak{S}_1$, and let all subgroups of $\mathfrak{G}$ conjugate to $\mathfrak{S}_1$ in $\mathfrak{G}$ already be conjugate to $\mathfrak{S}_1$ in $\mathfrak{S}$. Then $\mathfrak{G} = \mathfrak{B}\mathfrak{S}$, where $\mathfrak{B}$ is the normalizer of the subgroup $\mathfrak{S}_1$ in $\mathfrak{G}$.

§ 3. **Theorem.** Let m be the greatest Π-divisor of the order g of the group $\mathfrak{G}$. Then to every representation of the number m in the form of a product

$$m = h_1 h_2 \dots h_t,$$

where each of the factors $h_1, h_2, \dots, h_t$ is a certain reduced Π-integral divisor of the order g of the group $\mathfrak{G}$, there corresponds a representation of $\mathfrak{G}$ in the form of a product

$$\mathfrak{G} = \mathfrak{H}_1 \mathfrak{H}_2 \dots \mathfrak{H}_t,$$

where $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_t$ are certain pairwise permutable subgroups of $\mathfrak{G}$ such that the greatest Π-divisors of their orders are respectively the numbers $h_1, h_2, \dots, h_t$.

We shall give the main points of the proof of the theorem. Suppose that there exist groups for which the theorem is not fulfilled. Choose among such groups a group $\mathfrak{G}$ having the least order g. Then there exists a nonempty set of prime numbers Π and such a repre-

representation of the number m in the form $m = h_1 h_2 \dots h_t$ required by the theorem, to which the desired factorization of the group $\mathfrak{G}$ does not correspond. It follows from this that g is a Π-composite number and that $g > 1$ and $t > 1$.

Consider a chief series

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \dots \supset \mathfrak{G}_\lambda = \mathfrak{E} \tag{1}$$

of the group $\mathfrak{G}$ ($\mathfrak{E}$ is the identity subgroup).

Assigning $\mathfrak{E}$ to the special groups, we observe that if all terms of the series (1) were special, then $\mathfrak{G}$ itself would be special. But then $\mathfrak{G}$, obviously, would admit the required factorization. Therefore among the terms of (1) there exist nonspecial groups. Choose among them such a $\mathfrak{G}_\mu = \mathfrak{M}$ whose index μ is greatest. Since $\mathfrak{M}$, by assumption, is nonspecial, $\mathfrak{M} \neq \mathfrak{E}$, i.e. $\mu < \lambda$. Thus $\mathfrak{G}_{\mu+1} = \mathfrak{N}$ exists.

Denote the order of $\mathfrak{N}$ by n. Then the order of $\mathfrak{M}$ will have the form wn. Represent n in the form $n = n_1 n_2$, where n_2 is the greatest divisor of n relatively prime to w. Since $\mathfrak{N}$ is a special group, $\mathfrak{N} = \mathfrak{N}_1 \mathfrak{N}_2$, where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are subgroups of orders n_1 and n_2, respectively. But $(n_1, n_2) = 1$, and therefore $\mathfrak{N}_2$ will be invariant in $\mathfrak{G}$. Since $\mathfrak{N}_2$ is a special group and $(wn_1, n_2) = 1$, by the Schur-Zassenhaus theorem $\mathfrak{M}$ has a subgroup $\mathfrak{M}_1$ of order wn_1, and all subgroups of order wn_1 in $\mathfrak{M}$ are conjugate to $\mathfrak{M}_1$ in $\mathfrak{M}$.

1) $\mathfrak{M}_1$ is not invariant in $\mathfrak{G}$. Applying Lemma 6 to $\mathfrak{M}$ and $\mathfrak{M}_1$, we obtain $\mathfrak{G} = \mathfrak{V}\mathfrak{M}$, where $\mathfrak{V}$ of order v is the normalizer of $\mathfrak{M}_1$ in $\mathfrak{G}$. But $\mathfrak{M} = \mathfrak{M}_1 \mathfrak{N}_2$, whence $\mathfrak{G} = \mathfrak{V}\mathfrak{N}_2$. By Lemma 3, $v = v_1 v_2 \cdots v_t$, where $v_i = (v, h_i)$, will be the decomposition required by the theorem of the greatest Π-divisor of v. Since $\mathfrak{M}_1$ is not invariant in $\mathfrak{G}$, $v < g$, and the theorem holds for $\mathfrak{V}$. Consequently, $\mathfrak{V} = \mathfrak{F}_1 \mathfrak{F}_2 \dots \mathfrak{F}_t$, where the subgroups $\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_t$

are pairwise permutable, and the greatest Π-divisors of their orders are respectively equal to $v_1, v_2, \dots, v_t$.

Putting $\nu_i = (n_2, h_i)$, we verify on the basis of Lemma 3 that $\nu_1, \nu_2, \dots, \nu_t$ will be pairwise relatively prime Sylow divisors of n_2. But $\mathfrak{N}_2$ is a special group; hence we conclude that $\mathfrak{N}_2 = \mathfrak{N}'_1 \mathfrak{N}'_2 \dots \mathfrak{N}'_t$, where $\mathfrak{N}'_1, \mathfrak{N}'_2, \dots, \mathfrak{N}'_t$ are invariant subgroups of $\mathfrak{G}$ with pairwise relatively prime orders, and the greatest Π-divisors of their orders are respectively equal to $\nu_1, \nu_2, \dots, \nu_t$. But then $\mathfrak{G} = \mathfrak{V}\mathfrak{N}_2 = \mathfrak{H}_1 \mathfrak{H}_2 \dots \mathfrak{H}_t$, where $\mathfrak{H}_i = \mathfrak{F}_i \mathfrak{N}'_i$. The subgroups $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_t$ are, evidently, pairwise permutable, and with the help of Lemmas 2, 4, and 5 it is established that the greatest Π-divisors of their orders are respectively equal to $h_1, h_2, \dots, h_t$. Thus, for $\mathfrak{G}$ we have obtained a factorization of the desired form. A contradiction is obtained.

2) $\mathfrak{M}_1$ is invariant in $\mathfrak{G}$. The subgroup $\mathfrak{M} = \mathfrak{M}_1 \mathfrak{N}_2$ is a nonspecial normal divisor of $\mathfrak{G}$; the orders of $\mathfrak{M}_1$ and $\mathfrak{N}_2$ are relatively prime; $\mathfrak{N}_2$ is special and invariant in $\mathfrak{G}$. Therefore among the Sylow subgroups of $\mathfrak{M}_1$ there exists a subgroup $\mathfrak{R}$ not invariant in $\mathfrak{G}$. Denoting now by $\mathfrak{V}$ the normalizer of $\mathfrak{R}$ in $\mathfrak{G}$, we obtain, by Lemma 6, that $\mathfrak{G} = \mathfrak{V}\mathfrak{M} = \mathfrak{V}\mathfrak{M}_1\mathfrak{N}_2$.

Using the fact that $\mathfrak{M}_1/\mathfrak{R}$ is an elementary group, and that n_2 is the greatest divisor of n relatively prime to w, one can further show that the greatest Π-divisor of the order of $\mathfrak{M}_1$ will be a divisor of some number h_ρ from the series $h_1, h_2, \dots, h_t$. Since $\mathfrak{R}$ is not invariant in $\mathfrak{G}$, $\mathfrak{V} \neq \mathfrak{G}$, and the theorem holds for $\mathfrak{V}$. Hence, as in (1), we obtain the factorization

$$\mathfrak{G} = \mathfrak{F}_1 \mathfrak{F}_2 \dots \mathfrak{F}_t \mathfrak{M}_1 \mathfrak{N}'_2 \mathfrak{N}'_2 \dots \mathfrak{N}'_t = \mathfrak{H}_1 \mathfrak{H}_2 \dots \mathfrak{H}_t,$$

where

$$\mathfrak{H}_1 = \mathfrak{F}_1 \mathfrak{N}'_1, \dots, \mathfrak{H}_{\rho-1} = \mathfrak{F}_{\rho-1} \mathfrak{N}'_{\rho-1}, \mathfrak{H}_\rho = \mathfrak{F}_\rho \mathfrak{N}'_\rho \mathfrak{M}_1, \mathfrak{H}_{\rho+1} = \mathfrak{F}_{\rho+1} \mathfrak{N}'_{\rho+1}, \dots, \mathfrak{H}_t = \mathfrak{F}_t \mathfrak{N}'_t.$$

Applying again Lemmas 2, 4, and 5, we establish that the greatest Π-divisors of the orders of $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_t$ are respectively equal to $h_1, h_2, \dots, h_t$. We have obtained a contradiction.

The contradictions obtained prove the theorem.

Received
2 XII 1957

CITED LITERATURE

1. S. A. Chunikhin, *DAN*, **108**, No. 3, 397 (1956).
2. S. A. Chunikhin. *Matem. sborn.*, **43** (85), No. 1 (1957).

Note: Figure translations are in progress. See original paper for figures.