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V. A. ZALGALLER

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Abstract

Full Text

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V. A. ZALGALLER

ISOMETRIC EMBEDDING OF POLYHEDRA

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1. Let P^n be an n-dimensional polyhedron composed of simplexes of a space R^n of constant curvature. If the polyhedron P^n, consisting of simplexes s^k of various dimensions $k = 0, \dots, n$, is subdivided into simplexes $\bar{s}^k$, and each simplex $\bar{s}^k$ of the resulting polyhedron $\bar{p}$ is embedded in R^n (possibly with these simplexes overlapping one another), and in such a way that the points of these simplexes that are to be glued together in forming $\bar{P}^n$ coincide in position in R^n, then the gluing of these points may be regarded as carried out, and we say that the polyhedron P^n is isometrically embedded in R^n (possibly with self-intersections and overlaps). A simple example of such an embedding is given by a paper polygon folded several times and laid on a plane.

Theorem. *For $n = 1, 2, 3, 4$, every polyhedron P^n composed of simplexes of a space R^n of constant curvature can be isometrically embedded in R^n, if self-intersections and overlaps are allowed.*

2. We first explain the proposed construction by the example of embedding the development P^2 of a closed polyhedral surface in the plane E^2.

Mark on the surface P^2 the points A that are vertices of the development. Decompose P^2 into "Voronoi regions," including in each region $U(A)$ those points of the surface whose distance in P^2 from A is no greater than from any other of the marked points. The regions $\bar{U}(A)$ will have the form of geodesic polygons, each with one point A inside. In addition to the points common with other regions, assign to the boundary $\bar{U}(A)$ those points of this region that can be joined to A on P^2 by more than one shortest path. These are the points at which the region $U(A)$ "borders on itself." After this each region $\bar{U}(A)$ can be divided into triangles T lying on the surface, with vertices A and bases on the boundary $U(A)$, in such a way that adjacent triangles T with vertex at A will adjoin along whole lateral sides and, moreover, all the triangles T from all the regions will split into pairs of triangles with common bases. The triangles $T'T''$ adjoining along a common base will be isometric to equal plane triangles.

Retriangulate P^2 into the indicated triangles T. Separate each pair of triangles T', T'' with a common base from P^2, flatten it onto the plane, and fold it with respect to the common base. The result is a doubly covered plane triangle. We fold it once more with respect to the bisector of the angle at that vertex into which the original points A have fallen. In this process the original lateral sides of the triangles T', T'' will go from the common point A along one and the same ray. In the plane E^2 choose a point O and a ray l issuing from it. Place all the figures obtained above on E^2, putting the points A at O and directing the lateral sides of the triangles T along l. Under this embedding all gluings of the boundaries of T can be carried out, and P^2 turns out to be isometrically embedded in E^2.

The construction made applies to any developments specifying a metrized two-dimensional manifold with a polyhedral metric.

3. We pass to the theorem formulated above. We shall carry out the proof by induction on n; however, for each succeeding n we shall need a somewhat different construction of the auxiliary system of points $\{A\}$.

Suppose the theorem is true for every R^{n-1} with arbitrary P^{n-1}. Consider arbitrary R^n and P^n. For simplicity we shall assume that: 1) the polyhedron P^n is connected; 2) in P^n there are no simplexes s^k $(k < n)$ that are not faces of simplexes s^n from P^n; 3) at least two simplexes s^n are glued to each simplex s^{n-1} in P^n. Conditions 1) and 2) can be ensured by gluing on additional simplexes, and condition 3) by gluing, along the free s^{n-1}, a second copy of P^n. After the embedding in R^n, the added simplexes may be omitted.

By virtue of 1), the polyhedron P^n will be a metric space: it is enough to understand by the distance between its points the lower bound of the lengths of polygonal lines in P^n joining these points.

Mark in P^n a finite system of points $\{A\}$, including in it all vertices s^0 and also some set of points lying on faces of simplexes s^k $(k \leqslant n-2)$. After the choice of the system of points $\{A\}$, the metric space P^n is divided into "Voronoi regions" in the following way. To each chosen point A there is adjacent a finite number of simplexes s^n. In each such simplex we draw from the point A, in all directions interior to the simplex s^n, rays, and continue each ray as a geodesic in the space P^n as long as its points remain closer to the given point A than to any other point of the system $\{A\}$, and as long as this line remains the unique shortest one joining its points with A. If the geodesic being drawn reaches a face s^k $(0 < k \leqslant n-1)$, then it may branch, continuing in the simplexes s^n adjacent to s^k in directions interior to these simplexes. These open geodesic rays issuing from A fill a set of points which we shall call the Voronoi region $U(A)$ of the point A. The closures of such regions $\overline{U}(A)$ fill all of P^n.

We shall now make the assumption that the finite system of points $\{A\}$ has been chosen in such a way that none of the regions $U(A)$ intersects simplexes s^k $(k \leqslant n-2)$. Then the regions $\overline{U}(A)$ will meet along faces that are polyhedra. Indeed, in this case along each ray drawn there exists an n-dimensional neighborhood isometric to a region in R^n; therefore the boundary of adjacency of the regions $U(A)$, into which this ray abuts, locally turns out to be an $(n-1)$-dimensional plane in R^n.

If some ray passes through a face s^{n-1} and continues beyond it, branching in several simplexes s^n adjacent to s^{n-1}, then we shall regard the initial segment of such a ray in the corresponding number of copies. (Later these copies will again be identified.) Under the assumption made about the system $\{A\}$, each of the regions $\overline{U}(A)$ can be divided into a finite number of figures lying in P^n, isometric to n-dimensional pyramids in R^n. These pyramids will have vertices at the points A, and their bases will be $(n-1)$-dimensional polyhedra along which the regions $U(A)$ adjoin. Some of the pyramids with vertices at A may be filled by geodesic rays that coincide on the initial segments. In this case the pyramids as a whole also coincide near A up to some face s^{n-1}, and then, beyond this face, continue in different s^n. For the time being we regard these pyramids as distinct.

All the pyramids adjacent along one common base will be isometric to equal pyramids in R^n (they have equal bases and identical dist—

distances from the vertices to the corresponding points of the base). In view of property 3), at least two pyramids will be incident with each base.

We divide P^n into the indicated pyramids. Each group of pyramids with a common base will be separated from the composition of P^n and regarded as different copies of one and the same pyramid in R^n, but as copies with identified bases. Each of such "multilayered" pyramids can be placed with its vertex at the center of the sphere R^{n-1} in R^n, taking the radius of this sphere so small that the bases of all the pyramids lie outside R^{n-1}. Each pyramid cuts out on the sphere R^{n-1} a multiply covered spherical polygon, more precisely a polyhedron in R^{n-1}. We shall temporarily identify the copies of this polyhedron. In addition, to the polyhedra obtained in R^{n-1} we transfer the necessary rule for gluing the lateral faces of the pyramids in the composition of P^n. We obtain a polyhedron P^{n-1}, composed of polyhedra of the spherical space R^{n-1}.

By the induction hypothesis, the polyhedron P^{n-1} can be embedded isometrically in R^{n-1}. The placement of the pyramids corresponding to this embedding, with vertices at the center of the sphere R^{n-1}, makes it possible to carry out all the gluings of these pyramids required in the composition of P^n, including not only gluings along adjacent faces, but also, in the necessary cases, the identification of regions adjacent to a vertex of some pyramids. Thus P^n will be isometrically embedded in R^n.*

4. It remains for us to prove the theorem for $n = 1$ and, for subsequent n, to indicate a construction of the system $\{A\}$. For $n = 1$ it is enough to lay off each simplex s^1 in R^1 in folded-in-half form, placing the beginning and end of s^1 at one point O. Since all s^0 will be at O, the necessary gluings can be carried out, and P^1 will be embedded in R^1. For $n = 2$ it is enough to take as $\{A\}$ the set of vertices s^0.
5. Let $n = 3$, and let $2l$ be the least of the distances in P^3 between all possible simplices of P^3 having no common points. On each s^1 mark two points A_1 at equal distances $\varepsilon_1 < l$ from the ends of the simplex s^1. These

points divide every s^1 into end segments s_1^1 and middle segments s_2^1. On each s_2^1 mark an ε_2-net of points A_2. In the system $\{A\}$ include all points s^0, A_1, A_2. With an appropriate choice of $\varepsilon_1 > \varepsilon_2 > 0$, all regions $U(A)$ will not intersect the simplices s^1. Indeed, the smallness of ε_2 ensures that no point on s_2^1 can be closer to any point of the system $\{A\}$ than to those points of this system which form the ε_2-net on s_2^1. Suppose that a point $X \in s_1^1$ is closer to some point $\widetilde{A}$ than to the ends of s_1^1. Then, by the smallness of ε_1, the point $\widetilde{A}$ lies on a simplex $\tilde{s}^1$ having a common end s^0 with s_1^1. But then either $\widetilde{A}$ is the point $\widetilde{A}_1$ lying on $\tilde{s}^1$ near s^0, or $\widetilde{A}_1$ is still closer to X than $\widetilde{A}$. But in view of the position of the end A_1 of the segment s_1^1 and of the point $\widetilde{A}_1$ at the same distance from s^0, the point X cannot be closer to $\widetilde{A}$ than to A_1.

The theorem is proved for $n = 1, 2, 3$. We shall not give the somewhat more complicated construction of the system $\{A\}$ which proves the theorem for $n = 4$. At present we do not have a construction of the system $\{A\}$ for arbitrary n.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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* Even in the case when R^n is Euclidean and P^n is a manifold, our construction leads to the spherical R^{n-1} and to a general polyhedron P^{n-1}. Therefore the theorem was formulated at once for general polyhedra and spaces of constant curvature.

Note: Figure translations are in progress. See original paper for figures.