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V. A. YAKUBOVICH

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Abstract

Full Text

MATHEMATICS

V. A. YAKUBOVICH

ON THE DYNAMIC STABILITY OF ELASTIC SYSTEMS

(Presented by Academician V. I. Smirnov, 25 IV 1958)

An extensive technical literature is devoted to the dynamic stability of elastic systems ^(1,2). The first systematic exposition of the general theory was given in ⁽²⁾. Here we consider some mathematical questions arising in the general theory.

Let \mathfrak{M} be a class of matrices $A(t) \in L(0, 2\pi)$, $A(t + 2\pi) = A(t)$ almost everywhere, containing the constant matrix C , which is reducible to diagonal form with purely imaginary diagonal elements. Then all solutions of the “unperturbed system”

$$\frac{dx}{dt} = Cx \quad (1)$$

are bounded as $t \rightarrow \infty$.

A frequency $\theta > 0$ is called **resonant** for system (1) in the class \mathfrak{M} if, for every $\varepsilon > 0$, one can find a matrix $A(t) \in \mathfrak{M}$,

$$\int_0^{2\pi} \|A(t) - C\| dt < \varepsilon,$$

such that the “perturbed system”

$$\frac{dx}{dt} = A(\theta t)x \quad (2)$$

has solutions unbounded as $t \rightarrow \infty$.

A frequency $\theta_0 > 0$ is called **critical** for system (1) in the class \mathfrak{M} if, for any $\varepsilon > 0$, $\delta > 0$, one can find a matrix $A(t) \in \mathfrak{M}$,

$$\int_0^{2\pi} \|A(t) - C\| dt < \varepsilon,$$

and a number θ , $|\theta - \theta_0| < \delta$, for which system (2) has solutions unbounded as $t \rightarrow \infty$.

For applications, of special interest is the case where systems (1), (2) are of canonical form, and also the more particular case where the unperturbed system has the form

$$\frac{d^2 y}{dt^2} + P_0 y = 0,$$

where $P_0 = P_0^* > 0$, while the perturbed system has either the form

$$\frac{d^2 y}{dt^2} + P(\theta t) y = 0, \quad (3)$$

where the matrix $P(\tau) = P(\tau)^* = P(\tau + 2\pi)$ is specified only by belonging to some class, or the form

$$\frac{d^2 y}{dt^2} + [P_0 + \varepsilon Q(\theta t)] y = 0, \quad (4)$$

where $Q(\tau) = Q(\tau)^* = Q(\tau + 2\pi) \in L(0, 2\pi)$. Here P_0 , $P(\tau)$, $Q(\tau)$ are real matrices. In the last case the class $\mathfrak{M} = \mathfrak{M}_Q$ is a one-parameter family of matrices.

We shall restrict ourselves here to systems (3), (4), although all the results carry over to canonical systems and to some other systems of a more general form.

Let k be the order and let ω_j^2 be the eigenvalues of the matrix P_0 ($\omega_j > 0$ —the natural frequencies of the corresponding real system); $P_0 v_j = \omega_j^2 v_j$, $(v_j, v_j) = 1$, $\omega_{-j} = -\omega_j$ ($i = 1, \dots, k$). The numbers $\rho_j = \exp(2\pi\omega_j i/\theta)$ will, for the unperturbed system, be multipliers of the first kind when $j > 0$ and of the second kind when $j < 0$ (3).

M. G. Krein proved the following remarkable fact (3)*: in the class of all systems (3), the critical and resonance frequencies are the frequencies

$$\theta = (\omega_j + \omega_h)/m, \quad j, h = 1, \dots, k, \quad m = 1, 2, \dots$$

Using the results of (4), it is easy to obtain an analogous formula for the class of all systems of canonical form.

For applications, narrower classes of systems are of interest, for example the one-parameter class of systems (4).

In the plane $\{\varepsilon, \theta\}$, the set of points for which system (4) has solutions unbounded as $t \rightarrow \infty$ decomposes into regions of “dynamic instability” adjacent to points $(0, \theta_0)$, where θ_0 are the critical frequencies in the class \mathfrak{M}_Q .

Suppose that at the point

$$\rho_0 = \exp(2\pi i \beta_0 / \theta_0)$$

there coincide p multipliers, among which there are multipliers of different kind. Then p equalities

$$\omega_j = \beta_0 + m_j \theta_0, \quad j = j_1, \dots, j_p,$$

will hold, where m_j are integers. Define the p^2 numbers $\chi_{j,h}$, $j, h = j_1, \dots, j_p$, by the formula

$$\chi_{j,h} = \frac{1}{2\sqrt{|\omega_j \omega_h|}} (Q^{(m_j - m_h)} v_j, v_h),$$

where $Q^{(m)}$ are the Fourier coefficients of the matrix $Q(\tau)$,

$$Q(\tau) \sim \sum_{-\infty}^{\infty} Q^{(m)} e^{im\tau}.$$

In particular, if

$$\theta_0 = (\omega_{j_0} + \omega_{h_0})/m \neq (\omega_j + \omega_h)/m',$$

for triples of numbers $\{j, h, m\} \neq \{j_0, h_0, m'\}$, $j, j_0, h, h_0, m, m' > 0$, then $p = 2$ and

$$\begin{aligned} \chi_{l,l} &= \frac{1}{2|\omega_l|} (Q^{(0)} v_l, v_l), \quad l = j_0, -h_0, \\ \chi_{j_0, -h_0} &= \frac{1}{2\sqrt{\omega_{j_0} \omega_{h_0}}} (Q^{(m)} v_{j_0}, v_{h_0}). \end{aligned} \quad (5)$$

We shall call this case the general case, and the remaining ones special.

Theorem 1.** a) *In the general case the frequency*

$$\theta_0 = (\omega_{j_0} + \omega_{h_0})/m$$

is a resonance frequency if

$$\Delta = \frac{1}{4} (\chi_{j_0, j_0} + \chi_{-h_0, -h_0})^2 + |\chi_{j_0, -h_0}|^2 > 0,$$

and is not a resonance frequency if $\Delta < 0$. When $\chi_{j_0, -h_0} \neq 0$, the frequency

$$\theta_0 = (\omega_{j_0} + \omega_{h_0})/m$$

is critical.

The straight lines $\theta = \theta_0 + \chi^{(\pm)} \varepsilon$, where

$$\chi^{(\pm)} = \frac{1}{m} (\chi_{j_0, j_0} + \chi_{-h_0, -h_0} \pm 2|\chi_{j_0, -h_0}|), \quad (6)$$

are tangent to the boundaries of the region of dynamic instability. Moreover, the segment of the straight line $\theta = \theta_0 + \chi\varepsilon$, $\chi^{(-)} < \chi < \chi^{(+)}$, for sufficiently small ε lies in the region of dynamic instability.

* M. G. Krein calls a critical frequency what we here call a resonance frequency; the statement itself is formulated in (3) in somewhat different terms. (There is, however, no contradiction here, since in the class of all systems (3) the sets of critical and resonance frequencies coincide.) The definition of a critical frequency given here apparently corresponds to what is understood by this in applications.

** This theorem can apparently be proved by arguments close to (4). We proceeded here in another way, which made it possible to express the numbers $\chi_{j,h}$ through the Fourier coefficients of the perturbation $Q(\tau)$.

for sufficiently small $\varepsilon > 0$ lies in the region of dynamic instability.

b) In the special case, for each point ρ_0 of the unit circle at which multipliers of different kinds coincide, one should form the equation

$$\det \|\chi_{j,h} - \alpha\gamma_{j,h}\| = 0, \quad j, h = j_1, \dots, j_p,$$

where

$$\chi_{j,h} = 0 \quad \text{for } j \neq h, \quad \gamma_{j,j} = \text{sign } j.$$

This equation (of degree p) will have real coefficients.

If all the roots α_j of this equation are real and distinct, and this holds for every point ρ_0 , then the frequency θ_0 is not resonant. If for at least one point ρ_0 this equation has complex roots, then the frequency θ_0 is resonant.

The frequency θ_0 is critical if the equation

$$\det \|\chi_{j,h} - \chi\omega_j\gamma_{j,h} - \alpha\gamma_{j,h}\| = 0, \quad j, h = j_1, \dots, j_p, \quad (7)$$

has complex roots α_j for at least one χ .

In this case, the segment of the straight line $\theta = \theta_0 + \chi\varepsilon$, for the corresponding χ and sufficiently small ε , lies in the region of dynamic instability.

Thus, the values χ separating the intervals on which all roots of equation (7) are real and distinct, and the intervals on which there are complex roots, are the angular coefficients of tangents to the regions of dynamic instability.

Suppose that the general case holds. We shall call a region of instability adjoining the point $(0, \theta_0)$ **wide** if it contains a sufficiently small sector with vertex at $(0, \theta_0)$ ($\chi^{(+)} \neq \chi^{(-)}$), and **narrow** otherwise ($\chi^{(+)} = \chi^{(-)}$).

Corollary. If the m -th Fourier coefficient $Q^{(m)} = 0$, then for equation (4) all $k(k+1)/2$ regions of dynamic instability adjoining the points $(0, \theta_0)$, $\theta_0 = (\omega_j + \omega_h)/m$, $j, h = 1, \dots, k$, are narrow. If $Q^{(m)} \neq 0$, then among these regions there is at least one wide region*.

Example 1 ⁽²⁾, p. 311** . System (5), where

$$k = 2, \quad P_0 = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}, \quad Q(\tau) = -(\alpha_0 + \beta_0 \cos \tau) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the corollary to Theorem 1, all regions of dynamic instability corresponding to the frequencies $\theta = (\omega_j + \omega_h)/m$, $j, h = 1, 2$, $m > 1$, are narrow. After simple calculations we obtain that the frequencies $\theta = 2\omega_1$, $\theta = 2\omega_2$ correspond to narrow regions of dynamic instability, and the frequency $\theta = \omega_1 + \omega_2$ to a wide one. Instability occurs when

$$-\frac{\varepsilon\beta_0}{\sqrt{\omega_1\omega_2}} + \dots < \theta - (\omega_1 + \omega_2) < \frac{\varepsilon\beta_0}{\sqrt{\omega_1\omega_2}} + \dots$$

* As far as the author knows, up to now only regions of dynamic instability corresponding to the principal frequencies $\theta = 2\omega_j/m$ have been constructed (and it was not proved that instability actually occurs for the constructed regions—they were only regions “suspected” of dynamic instability). These regions turned out to be narrow. It was believed that “combination resonance” $\theta = (\omega_j + \omega_h)/m$, $j \neq h$, is of secondary importance. As follows from what has been set out above, in such cases the principal role is played not by the “principal” resonance, but by the “combination” resonance.

** In the notation of ⁽²⁾, $\alpha_0 = \alpha a_{12}\omega_1^2 = \alpha a_{21}\omega_2^2$, $\beta_0 = \beta a_{12}\omega_1^2$. Note, with reference to book ⁽²⁾, that many problems of dynamic stability of plates and plane bending form lead to an equation of this type.

It can be shown (see also Theorem 2) that all frequencies $\theta = (\omega_j + \omega_h)m$ are critical.

Example 2. System (4), where

$$P_0 = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}, \quad Q = \begin{pmatrix} \alpha(\tau) & \beta(\tau) \\ \beta(\tau) & \gamma(\tau) \end{pmatrix}.$$

Only the mean values $\alpha_{av}, \beta_{av}, \gamma_{av}$ and the deviations $\alpha_0, \beta_0, \gamma_0$ from the mean values are known: $|\alpha(\tau) - \alpha_{av}| \leq \alpha_0$, $|\beta(\tau) - \beta_{av}| \leq \beta_0$, $|\gamma(\tau) - \gamma_{av}| \leq \gamma_0$ (this defines the class \mathfrak{M}). Carrying out simple calculations, we obtain:

	$\frac{2\omega_1}{m}$	$\frac{2\omega_2}{m}$	$\frac{\omega_1 + \omega_2}{m}$
Frequency			
Condition for the frequency to be critical	$\alpha_0 \neq 0$	$\gamma_0 \neq 0$	$\beta_0 \neq 0$
Condition for the frequency to be resonant	$ \alpha_{av} < \alpha_0$	$ \gamma_{av} < \gamma_0$	$\left \frac{\alpha_{av}}{\omega_1} + \frac{\gamma_{av}}{\omega_2} \right < \frac{2\beta_0}{\sqrt{\omega_1\omega_2}}$

Moreover, if in the last conditions the sign of the inequality is reversed, then the corresponding frequency is not resonant.

Theorem 2. a) For system (4), the boundary of any region of dynamic instability can be specified by an equation $\theta = \theta(\varepsilon)$, where $\theta(\varepsilon)$ is an analytic function of ε at the point $\varepsilon = 0$.

b) Suppose the general case occurs, $\theta_0 = (\omega_j + \omega_h)/m$, and for $\theta = \theta_1(\varepsilon)$, $\theta = \theta_2(\varepsilon)$, where $\theta_1(0) = \theta_2(0) = \theta_0$, $\theta_1(\varepsilon) < \theta_2(\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$, system (4) has multiple multipliers of different kinds. Then, for sufficiently small ε , the region $\theta_1(\varepsilon) < \theta < \theta_2(\varepsilon)$ is a region of dynamic instability.

The second part of the theorem shows that practically every frequency $\theta_0 = (\omega_j + \omega_h)/m$ is critical in the narrowest one-parameter class of systems (4).

Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.

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