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Abstract

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MATHEMATICS

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CONJUGATE LYAPUNOV NORMS

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In the work of Yu. S. Bogdanov ([1]) an abstract definition of a Lyapunov norm (λ-norm) in an arbitrary linear space is given; a particular example is the space of solutions of a system of linear homogeneous differential equations with a characteristic exponent as the λ-norm. Such an abstract approach turns out to be especially useful in the study of the characteristic exponents of conjugate systems of differential equations, making it possible to discover a number of new facts.

With applications in mind, we shall restrict ourselves to considering n-dimensional Euclidean spaces and norms taking real numerical values.

I. Notation and definitions. $E^n = \{x\}$ and $\overline{E}^n = \{z\}$ are conjugate Euclidean spaces; (x, z) is the scalar product (bilinear functional); $X = x_1, x_2, \dots, x_n$ is a basis in E^n; $Z = z_1, z_2, \dots, z_n$ is a basis in $\overline{E}^n$ (arbitrary); if

$$(x_i, z_j) = \delta_{ij}, \tag{1}$$

then (X, Z) is a conjugate pair of bases; $\omega(x)$ is a λ-norm in E^n; $\gamma(z)$ is a λ-norm in $\overline{E}^n$.

The norms ω and γ are called **conjugate** if, for all pairs of conjugate bases, from the condition

$$(x_i, z_i) = 1 \tag{2}$$

it follows that

$$\omega(x_i) + \gamma(z_i) \geq 0, \qquad i = 1, 2, \dots, n. \tag{3}$$

In what follows only conjugate norms will be considered.

The **defect** of a conjugate pair of bases (X, Z) is the number

$$\mu_{xz} = \max_i [\omega(x_i) + \gamma(z_i)],$$

and the minimum of μ_{xz} over all possible conjugate pairs (X, Z) will be called, following D. M. Grobman ([2]), the **irregularity coefficient** μ of the given pair of norms (ω, γ).

It follows from (3) that $\mu \geq 0$; the pair of norms (ω, γ) is called **regular** if $\mu = 0$.

If X is a λ-basis ([1]), then it is convenient to choose the numbering $x_1, x_2, \dots, x_n$ so that

$$\omega_1 \leq \omega_2 \leq \dots \leq \omega_n, \qquad \omega_i = \omega(x_i), \tag{4}$$

and among these numbers to single out only the distinct ones:

$$\Omega_1 < \Omega_2 < \dots < \Omega_p, \qquad p \leq n, \tag{4'}$$

endowing them with "multiplicities" $n_1, n_2, \dots, n_p$, so that $\omega_1 = \omega_2 = \dots = \omega_{n_1} = \Omega_1$, $\omega_{n_1+1} = \omega_{n_1+2} = \dots = \omega_{n_1+n_2} = \Omega_2$, and so on.

As shown in (1), the numbers (4′) and their multiplicities are invariants of λ-bases.

For the norm $\gamma(z)$ and its λ-basis Z, however, it is more convenient to adhere to the reverse numbering:

$$\gamma_1 \geqslant \gamma_2 \geqslant \dots \geqslant \gamma_n, \qquad \gamma_i = \gamma(z_i); \tag{5}$$

$$\Gamma_1 > \Gamma_2 > \dots > \Gamma_q, \qquad q \leqslant n, \tag{5'}$$

with "multiplicities" m_i, $m_1 + m_2 + \dots + m_q = n$.

The Perron number, under the numberings (4), (5), is the number

$$\nu = \max_i (\omega_i + \gamma_i).$$

II. The following theorems hold:

Theorem 1. For given norms $\omega(x)$ and $\gamma(z)$ there always exists a binormal pair of conjugate bases (X, Z), i.e. one such that X is a λ-basis in E^n with respect to $\omega(x)$ and, simultaneously, Z serves as a λ-basis in $\overline{E}^n$ with respect to $\gamma(z)$.

Theorem 2. The distribution of vectors according to their norms is invariant for all binormal pairs in the following sense: choose one number from each of (4′) and (5′): Ω_i and Γ_k; then the number of those indices r for which $(x_r, z_r) = 1$ and $\omega(x_r) = \Omega_i$, while $\gamma(z_r) = \Gamma_k$, is the same in all binormal pairs.

Theorem 3. The defect of an arbitrary conjugate pair (X, Z) is equal to the irregularity coefficient whenever at least one of the two bases—X or Z—is a λ-basis in its space.

In particular:

Theorem 4. The defect of a binormal pair coincides with the irregularity coefficient.

Theorem 5. The Perron number and the irregularity coefficient are in the relation

$$0 \leqslant \nu \leqslant \mu \leqslant n\nu.$$

In particular:

Theorem 6. A pair of norms $(\omega; \gamma)$ is regular if and only if $\nu = 0$, i.e. when, under the numberings (4), (5), we have

$$\gamma_i = -\omega_i, \qquad i = 1, 2, \dots, n.$$

The proofs of these theorems are, for the most part, purely geometric and are based on considering the possible mutual arrangements of the hyperplanes M_i (1) in the spaces E^n and $\overline{E}^n$.

III. Let there be given: a system of differential equations

$$X' = A(t)X \tag{6}$$

(A, X are $n \times n$ matrices) and the conjugate system

$$Z' = -A^*(t)Z. \tag{7}$$

It is known that if X is a fundamental matrix for (6), and $Z = X^{*-1}$, i.e.

$$X^* Z = I, \tag{8}$$

then Z is a fundamental matrix for (7). If the columns of the matrices X and Z are regarded as vectors and called bases, then relation (8) is equivis equivalent to (1), so that (2) also holds. By Cauchy' s inequality, hence $|x_i|\,|z_i| \geqslant 1$. In this case, passing to the characteristic exponents (taking $\lim_{t\to\infty} t^{-1} \ln|\ |$), we obtain

$$\text{exponent } x_i + \text{ exponent } z_i \geqslant 0,$$

which means that the conjugacy condition (3) is satisfied. And since the exponents are λ-norms, all the preceding theory carries over without change to the case under consideration: Theorems 1 and 2 become theorems on the existence and construction of conjugate fundamental systems that are simultaneously normal in the sense of A. M. Lyapunov; Theorems 3 and 4 clarify the meaning of D. M. Grobman' s coefficient of irregularity, while Theorems 5 and 6 become O. Perron' s refined theorem [3] on regular systems.

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[3] O. Perron, *Math. Zs.*, **31**, 1, 159 (1929).

Note: Figure translations are in progress. See original paper for figures.