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Abstract

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MATHEMATICS

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ON THE BEST APPROXIMATION OF CLASSES OF FUNCTIONS REPRESENTABLE IN THE FORM OF A CONVOLUTION

(Presented by Academician N. N. Bogolyubov on 29 VI 1957)

1. Let H_p $(1 \leq p < \infty)$ be the class of real functions $\varphi(t) \in L^p$ with period ω, satisfying the condition

$$\|\varphi\| = \left\{\int_0^\omega |\varphi(t)|^p dt\right\}^{1/p} \leq 1,$$

and let H_∞ be the class of essentially bounded measurable functions $\varphi(t)$ with period ω, for which

$$\operatorname{ess\,sup} |\varphi(t)| \leq 1.$$

Suppose that the function $f(x)$ admits a representation in the form of a convolution:

$$f(x) = \frac{1}{\omega}\int_0^\omega K(t-x)\varphi(t)\,dt \quad (\varphi(t) \in H_p), \tag{1}$$

where $K(t)$ is a function with period ω, and, in the case $1 < p \leq \infty$, $K(t) \in L^q$

$$\left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

while in the case $p = 1$, $K(t)$ is continuous. The totality of all functions of the form (1) will be denoted by $C(K, H_p)$.

Here we consider the problem of the best approximation of the class of functions $f(x)$, representable in the form (1), by polynomials in a given Chebyshev system ([1]) of continuous functions with period ω, $\{f_k(x)\}$ $(k = 1, 2, \ldots, n)$; general formulas are established for the quantity

$$M_n^{(p)} = \sup_{\varphi \in H_p} E_n(f)_C, \qquad 1 \leq p \leq \infty, \tag{2}$$

where

$$E_n(f)_C = \min_{\alpha_k} \max_{0 \le x \le \omega} \left| f(x) - \sum_{k=1}^{n} \alpha_k f_k(x) \right|$$

(α_k are real numbers) is the best approximation of the function $f(x)$ by polynomials in the system $\{f_k(x)\}$; the properties of extremal functions $f_0(x)$, for which $E_n(f_0)_C = M_n^{(p)}$, are investigated, and the relations between the quantity $M_n^{(p)}$ and the best approximation of the corresponding kernel $K(t)$ in the metric L^q are studied.

For the case $p = \infty$, this problem was studied earlier by Favard ([2]), Akhiezer and Krein ([3]), Nadem ([4]), Nikol' skii ([5]), and others.

2. Let

$$0 \le x_1 < x_2 < \cdots < x_{n+1} < \omega.$$

Put ([6])

$$M_i = \begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{i-1}) & f_1(x_{i+1}) & \cdots & f_1(x_{n+1}) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_{i-1}) & f_2(x_{i+1}) & \cdots & f_2(x_{n+1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n(x_1) & f_n(x_2) & \cdots & f_n(x_{i-1}) & f_n(x_{i+1}) & \cdots & f_n(x_{n+1}) \end{vmatrix},$$

$$M = \sum_{i=1}^{n+1} M_i, \qquad a_i = (-1)^i \frac{M_i}{M} \quad (i = 1, 2, \dots, n+1).$$

By $a_i^{(0)}$ we shall denote the value of a_i for $x_j = x_j^{(0)}$ $(j = 1, 2, \dots, n+1)$, and by $\{x_i^{(0)}\}$ a system of points

$$0 \le x_1^{(0)} < x_2^{(0)} < \cdots < x_{n+1}^{(0)} < \omega.$$

In what follows, by a Chebyshev alternant of the function $f_0(x)$ we shall mean a system of points $\{x_i^{(0)}\}$ at which the difference between the given function and its polynomial of best approximation assumes the maximal values $E_n(f_0)_c$ with alternating signs.

3. The case $1 < p \le \infty$.

Theorem 1. Let $K(t) \in L^q \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$.

Then:

$$1^\circ. \quad M_n^{(p)} = \frac{1}{\omega} \max_{0 < x_1 < \cdots < x_{n+1} < \omega} \left\{ \int_0^\omega \left| \sum_{i=1}^{n+1} a_i K(t - x_i) \right|^q dt \right\}^{1/q}. \tag{3}$$

2°. There exists a function $f_0(x) \in C(K, H_p)$ for which $E_n(f_0)_c = M_n^{(p)}$.

3°. The maximum in formula (3) is attained for those and only those systems of points $\{x_i^{(0)}\}$ which are Che

Generally speaking, the extremal function $f_0(x)$ is not unique. In the case $1 < p < \infty$, each extremal function $f_0(x)$ is determined by a unique function $\varphi_0(t) \in H_p$. In order that, for $M_n^{(p)} > 0$, the function $\varphi_0(t) \in H_p$ determine an extremal function $f_0(t)$, it is necessary and sufficient that there exist a system of points $\{x_i^{(0)}\}$ such that

$$M_n^{(p)} = \frac{1}{\omega} \left\{ \int_0^\omega \left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right|^q dt \right\}^{1/q}$$

and, almost everywhere,

$$\varphi_0(t) = \left\{ \frac{1}{\omega M_n^{(p)}} \right\}^{q-1} \left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right|^{q-1} \operatorname{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

on the interval $0 \leq t \leq \omega$.

The extremal function $f_0(x)$ is unique if and only if all the functions

$$\left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right| \operatorname{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

coincide almost everywhere on $[0, \omega)$ for all systems of points $\{x_i^{(0)}\}$ for which the maximum in formula (3) is attained.

In the case $p = \infty$, one and the same extremal function $f_0(x)$ may be determined by different functions $\varphi_0(t) \in H_\infty$. In order that the function $\varphi_0(t) \in H_\infty$ determine an extremal function $f_0(x)$, it is necessary and sufficient that there exist a system of points $\{x_i^{(0)}\}$ such that

$$M_n^{(\infty)} = \frac{1}{\omega} \int_0^\omega \left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right| dt,$$

$$\varphi_0(t) = \operatorname{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

almost everywhere on the set

$$\mathcal{E}\left\{t : \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \neq 0\right\}, \qquad \operatorname{mes}\mathcal{E} > 0.$$

For there to exist a unique function $\varphi_0(t) \in H_\infty$ determining the extremal function $f_0(x)$, it is necessary and sufficient that all the functions

$$\operatorname{sign}\left\{\sum_{i=1}^{n+1} a_i^{(0)} K\left(t - x_i^{(0)}\right)\right\}$$

coincide almost everywhere on $[0,\omega)$ for all systems of points $\{x_i^{(0)}\}$ for which the maximum in formula (3) is attained.

In the following two theorems we shall assume that $K(t) \in L^q[0,2\pi)$ $(1 \le q < \infty)$; $\varphi(t) \in H_p[0,2\pi)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$; $T_n(t)$ is a trigonometric polynomial of degree not exceeding n, and

$$E_n(f)_c = \min_{T_{n-1}} \|f(x) - T_{n-1}(x)\|_c,$$

where $f(x)$ is an arbitrary continuous function with period 2π.

Theorem 2. Let $K(t) \in L^q[0,2\pi)$ $(1 \le q < \infty)$ and

$$f(x) = \frac{1}{\pi}\int_0^{2\pi} K(t - x)\varphi(t)\,dt, \tag{4}$$

where $\varphi(t) \in H_p[0,2\pi)$. Then

$$\sup_{\varphi\in H_p} E_n(f)_c \le \frac{1}{\pi}\min_{T_{n-1}}\left\{\int_0^{2\pi} |K(t) - T_{n-1}(t)|^q\,dt\right\}^{1/q}, \tag{5}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $1 < p < \infty$, then in order that inequality (5) become an equality, it is necessary and sufficient that there exist a system of points

$$0 \le x_1^{(0)} < \cdots < x_{2n}^{(0)} < 2\pi$$

such that:

$$1^\circ. \qquad \sup_{\varphi\in H_p} E_n(f)_c = \frac{1}{\pi}\left\{\int_0^{2\pi}\left|\sum_{i=1}^{2n} \tilde{a}_i^{(0)} K\left(t - x_i^{(0)}\right)\right|^q dt\right\}^{1/q}, \tag{6}$$

where

$$\tilde{a}_i^{(0)} = (-1)^i \frac{Q_i^{(0)}}{Q^{(0)}}, \qquad Q^{(0)} = \sum_{i=1}^{2n} Q_i^{(0)}, \qquad Q_i^{(0)} = \prod_{p,q \neq i} \sin \frac{x_p^{(0)} - x_q^{(0)}}{2}, \tag{7}$$

$$2n \geq p > q \geq 1, \qquad p, q \neq i, \qquad i = 1, 2, \dots, 2n.$$

$$2^\circ. \qquad \operatorname{sign}\{K_n(t - x_i^{(0)})\} \cdot \operatorname{sign}\{K_n(t - x_{i+1}^{(0)})\} \leq 0$$

almost everywhere on the interval $0 \leq t \leq 2\pi$, $i = 1, 2, \dots, 2n - 1$.

3°. All the functions $\left|K_n(t - x_i^{(0)})\right|$ are proportional on the interval $0 \leq t \leq 2\pi$, $i = 1, 2, \dots, 2n$, where

$$K_n(t) = K(t) - T_{n-1}^*(t),$$

and $T_{n-1}^*(t)$ is the polynomial of best approximation to the function $K(t)$ of degree $n - 1$ in L^q.

If, however, $p = \infty$, then for (5) to become an equality it is necessary that there exist a system of points

$$0 \leq x_1^{(0)} < x_2^{(0)} < \dots < \dots < x_{2n}^{(0)} < 2\pi$$

such that:

$$1^\circ. \qquad \sup_{\varphi \in H_\infty} E_n(f)_c = \frac{1}{\pi} \int_0^{2\pi} \left| \sum_{i=1}^{2n} \tilde{a}_i^{(0)} K\left(t - x_i^{(0)}\right) \right| dt. \tag{8}$$

$$2^\circ. \qquad \operatorname{sign}\{K_n(t - x_i^{(0)})\} \cdot \operatorname{sign}\{K_n(t - x_{i+1}^{(0)})\} \leq 0$$

almost everywhere on the interval $0 \leq t \leq 2\pi$, $i = 1, 2, \dots, 2n - 1$, where $K_n(t) = K(t) - T_{n-1}^*(t)$, and $T_{n-1}^*(t)$ is any polynomial of best approximation in the mean to the function $K(t)$ of order $n - 1$.

If these conditions are satisfied for some trigonometric polynomial $T_{n-1}^*(t)$ of best approximation in the mean to the function $K(t)$ of order $n - 1$, then (5) turns into the equality (5′).

Theorem 3. If $K(t) \in L$, $K(t + 2\pi) = K(t)$, $\varphi(t)$ is any continuous function with period 2π,

$$f(x) = \frac{1}{\pi} \int_0^{2\pi} K(t - x) \varphi(t) \, dt, \tag{9}$$

then

$$E_n(f)_C \le \frac{1}{\pi}\left\{\max_{0<x_1<\cdots<x_{2n}<2\pi}\int_0^{2\pi}\left|\sum_{i=1}^{2n}\tilde a_i K(t-x_i)\right|dt\right\}E_n(\varphi)_C, \qquad (10)$$

where $\tilde a_i$ is a function of the parameters x_j $(1\le j\le 2n,\ j\ne i)$, defined by (7). The inequality (10) cannot be improved.

Corollary. Let $f(x)$ be a periodic function with period 2π, having a continuous r-th derivative $f^{(r)}(x)$; then

$$E_n(f)_C \le \frac{K_r}{n^r}E_n(f^{(r)})_C, \qquad n=1,2,\ldots \qquad (11)$$

If, moreover, the conjugate function $\tilde f(x)$ has a continuous r-th derivative $\tilde f^{(r)}(x)$, then

$$E_n(f)_C \le \frac{\widetilde K_r}{n^r}E_n(\tilde f^{(r)})_C, \qquad n=1,2,\ldots \qquad (12)$$

Here $K_r,\ \widetilde K_r$ are the Favard constants ([3],[7]). The constants cannot be improved.

4. We now consider the case $p=1,\ K(t)\in C[0,\omega]$. Let H_V be the class of functions $g(t)$ of bounded variation on $[0,\omega]$ with norm $\|g\|_V=\int_0^\omega|dg|\le 1$. Put

$$F(x)=\frac{1}{\omega}\int_0^\omega K(t-x)\,dg(t). \qquad (13)$$

Theorem 4. If $K(t)\in C[0,\omega]$, then

$$M_n^{(1)}=\max_{g\in H_V}E_n(F)_C=\frac{1}{\omega}E_n(K)_C. \qquad (14)$$

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REFERENCES

1. N. I. Akhiezer, *Lectures on the Theory of Approximation*, Moscow-Leningrad, 1947.

2. J. Favard, Bull. des Sci. Math., 61, 209, 243 (1937).

3. A. I. Akhiezer, M. G. Krein, DAN, 15, 107 (1937).

4. B. Nagy, Ber. d. Math.-phys. Kl. Akad. d. Wiss. Leipzig, 90, 103 (1938).

5. S. M. Nikol' skii, Izv. AN SSSR, ser. matem., 10, 207 (1946).

6. E. Ya. Remez, *General Computational Methods of Chebyshev Approximation*, 1957.

7. S. B. Stechkin, Izv. AN SSSR, ser. matem., 20, 197 (1956).

Note: Figure translations are in progress. See original paper for figures.