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Abstract

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ON POLYHEDRAL SPECTRA AND THE DIMENSION OF BICOMPACTA, IN PARTICULAR BICOMPACT GROUPS

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Introduction. Below, unless otherwise stated, we consider spectra (for the definition see (^{1}), § 6) with projections "onto."

We shall call a spectrum $S = \{X_\alpha,\ \omega_\alpha^\beta\}$ **polyhedral** if the X_α are polyhedra; if, in addition, the projections ω_α^β are, under certain subdivisions of X_β and X_α, simplicial and affine on the simplexes of the subdivision, then the spectrum S will be called **simplicial**. A spectrum $S = \{X_\alpha,\ \omega_\alpha^\beta\}$ will be called ***n*-dimensional in the sense of** dim if $\dim X_\alpha \leq n$ for all α.

The main results of this paper are the following propositions:

Theorem 1. a) *Every bicompactum is the space of some polyhedral spectrum.*

b) *For zero-dimensional bicompacta this spectrum may be taken to be zero-dimensional, simplicial, and with projections "onto."*

Theorem 2. a) *If a bicompactum X is the space of an n-dimensional in the sense of* dim *spectrum $S = \{X_\alpha,\ \omega_\alpha^\beta\}$, then $\dim X \leq n$.*

b) *If a bicompactum X is the space of an n-dimensional polyhedral spectrum, then for any closed set $\Phi \subseteq X$ there exist arbitrarily close neighborhoods whose boundary belongs to the space of some $(n-1)$-dimensional spectrum of compacta.*

Along the way it is proved that:

c) *If $\dim X = n$, where $n = 0$ or 1, and X is the space of an n-dimensional polyhedral spectrum, then*

$$n = \dim X = \operatorname{ind} X = \operatorname{Ind} X.$$

It follows from this that the bicompacta of Lunc (^{2}) and Lokucievskii (^{3}), for which $\dim X = 1$, $\operatorname{ind} X = 2$, are not spaces of any one-dimensional polyhedral spectrum; i.e., generally speaking, the dimension $\dim X$ of a bicompactum X

cannot be defined as the least of the dimensions of the polyhedra approximating this bicompactum.

Theorem 3. *If a bicompactum X is the space of an n-dimensional simplicial spectrum, then $n \geq \operatorname{Ind} X$.*

Hence it follows: *if $\dim X = n$ and X is the space of an n-dimensional simplicial spectrum, then*

$$n = \dim X = \operatorname{ind} X = \operatorname{Ind} X.$$

In conclusion it is proved that:

Theorem 4. *For a bicompact topological group X*

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X.$$

§ 1. **Proof of Theorem 1.** a) Let τ be the weight of the bicompactum X; then X may be regarded as embedded in the Tychonoff cube J^τ of weight τ. Let us supply all one-dimensional faces J^τ with indices from some set $A = \{\alpha\}$. By $J^{n(\alpha_1 \cdots \alpha_n)}$ we shall denote the face of J^τ equal to $\prod_{i=1}^n J^{1(\alpha_i)}$, and by $X^{n(\alpha_1 \cdots \alpha_n)}$ the projection of the set X onto this face. Consider such subdivisions of the n-dimensional faces J^τ into cubes that the length of the one-dimensional faces of the cubes of the subdivision is equal to $1/2^n$. From $J^{n(\alpha_1 \cdots \alpha_n)}$ discard successively (in order of dimension) the open senior cubes that contain no points of $X^{n(\alpha_1 \cdots \alpha_n)}$. Taking the bordering polyhedra that remain after the operation of discarding, with the natural order and projections, we obtain the required spectrum.

b) If the bicompactum X is zero-dimensional, then in it one can choose a base of open-closed sets $\gamma = \{\Gamma_\alpha\}$, and define an embedding system of functions as follows: $f_\alpha = 0$ on Γ_α, $f_\alpha = 1$ on $X \setminus \Gamma_\alpha$. In this case the $X^{n(\alpha_1 \cdots \alpha_n)}$ will consist simply of a finite number of points and will give us the required spectrum.

Proof of Theorem 2. By definition, a base $\mathfrak{B}$ in the space X of the spectrum $S = \{X_\alpha, \mathfrak{D}_\alpha^\beta\}$ is defined as follows. A finite number of indices $\alpha_1, \ldots, \alpha_s$ is fixed; arbitrary open sets $V_{\alpha_1}, \ldots, V_{\alpha_s}$ in $X_{\alpha_1}, \ldots, X_{\alpha_s}$ are chosen, and the set

$$O = \mathcal{E}\{x,\ x = \{x_\alpha\}, \quad x_{\alpha_i} \in V_{\alpha_i}, \quad i = 1, \ldots, s\}$$

is declared to be an element of the base of the space X, induced by the collection $V_{\alpha_1}, \ldots, V_{\alpha_s}$. We first prove two lemmas.

Lemma 1. *Each element O of the base $\mathfrak{B}$ of the space X may be assumed to be induced by only one V_α.*

Indeed, let O be defined by the collection $V_{\alpha_1}, \dots, V_{\alpha_s}$. Taking in X_α, where $\alpha > \alpha_i,\ i = 1, \dots, s$, the set $\bigcap(\mathfrak{D}^{\alpha}_{\alpha_i})^{-1}V_{\alpha_i}$, we obtain the required V_α.

Lemma 2. *For any finite covering $\gamma = \{O_1, \dots, O_s\}$, consisting of elements of the base $\mathfrak{B}$, of a closed set $\Phi \subseteq X$, one can find X_α and in it a system of open sets $\gamma_\alpha = \{V_{\alpha_1}, \dots, V_{\alpha_s}\}$ inducing in X the system γ.*

Indeed, we may assume that O_i is induced by the set V_{α_i}, $i = 1, \dots, s$. Taking in X_α, $\alpha > \alpha_i$, $V_{ai} = (\mathfrak{D}^{\alpha}_{\alpha_i})^{-1}V_{\alpha_i}$, $i = 1, \dots, s$, we obtain the desired system γ_α.

Remark to Lemma 2. It is clear that $V_\alpha = \bigcup V_{\alpha i}$ induces in X the element of the base $O = \bigcup O_i$, containing the set Φ.

Let us now prove assertion a) of Theorem 2. Take an arbitrary covering of the bicompactum X. Inscribe in it a finite covering γ from elements of the base $\mathfrak{B}$. By Lemma 2 we find the corresponding X_α and its covering γ_α. Since $\dim X_\alpha \le n$, we inscribe in γ_α a covering ω_α of multiplicity $\le n + 1$, which induces in X a covering ω of multiplicity $\le n + 1$, inscribed in γ. Item a) is proved.

For the proof of item b) we preface two further lemmas.

Lemma 3. *A bicompactum X that is the space of a zero-dimensional spectrum of bicompacta is zero-dimensional.*

This follows from item a) of Theorem 2.

Lemma 4. *Let X be the space of the spectrum $S = \{X_\alpha, \mathfrak{D}^{\beta}_{\alpha}\}$; then X is the space of the spectrum $S' = \{X_\alpha, \mathfrak{D}^{\beta}_{\alpha}\}$, where $\alpha \ge \alpha_0$.*

The proof of this lemma presents no difficulties.

We now prove assertion b) of Theorem 2. X is the space of an n-dimensional polyhedral spectrum $S = \{X_\alpha, \wp^{\beta}_{\alpha}\}$. Since $\Phi \subset X$ is a bicompactum, by the remark to Lemma 2 one may assume that an arbitrary neighborhood $O\Phi$ is induced by an element V_{α_0}. By Lemma 4, we now pass to the consideration of the spectrum S', $\alpha \geqslant \alpha_0$.

For each α take the sets $F_\alpha = [V_\alpha] \setminus V_\alpha$, where $V_\alpha = (\wp^{\alpha}_{\alpha_0})^{-1}V_{\alpha_0}$. The sets F_α form a spectrum of compacta, if they are taken with the natural projections and order. It is enough to show that $\wp^{\beta}_{\alpha}F_\beta \subset F_\alpha$. But this follows easily from the transitivity of the spectrum. Take an arbitrary subdivision $X_{\alpha'}$; from the fact that F_α is nowhere dense in each n-dimensional simplex of the subdivision, it follows that $\dim F_\alpha \leqslant n - 1$.

We denote the space of the spectrum $s = \{F_\alpha, \wp^{\beta}_{\alpha}\}$ by F. We shall show that $[O\Phi] \setminus O\Phi \subseteq F$. Let $x = \{x_\alpha\} \in [O\Phi] \setminus O\Phi$. Every neighborhood Ox contains points of $O\Phi$, in particular also when Ox is a basic neighborhood induced by an arbitrary $V_{x_\alpha} \subseteq X_\alpha$, i.e. $V_\alpha \cap V_{x_\alpha} \ne \Lambda$, i.e. $x_\alpha \in F_\alpha$ for every α. Assertion b) is proved.

We prove part c). For bicompacta it is known that $\dim X \leqslant \operatorname{ind} X \leqslant \operatorname{Ind} X$, and for zero-dimensional bicompacta all these three dimensions coincide.

For $n = 0$ assertion c) is obvious. Let $n = 1$. Then every closed set $\Phi \subset X$ has an arbitrarily close neighborhood whose boundary belongs to a zero-dimensional bicompactum, i.e. is itself zero-dimensional; hence, $\operatorname{Ind} X \leqslant 1$.

§ 2. **Proof of Theorem 3.** Let the bicompactum X be the space of an n-dimensional simplicial spectrum $S = \{X_\alpha, \wp_\alpha^\beta\}$. We shall prove that $\operatorname{Ind} X \leqslant n$. Take an arbitrary closed set $\Phi \subset X$. Let $O\Phi$ be a neighborhood of Φ; one may assume that it is induced by some set V_{α_0}. Consider now the spectrum $S' = \{X_\alpha, \wp_\alpha^\beta\}$, $\alpha \geqslant \alpha_0$, and let Φ_α be the projection of Φ in X_α. Then $\Phi_{\alpha_0} \subseteq V_{\alpha_0}$. One can find a polyhedral neighborhood W_{α_0} of the set Φ_{α_0}, contained in V_{α_0}. Consider the sets $P_\alpha = [W_\alpha] \setminus W_\alpha$, where $W_\alpha = (\wp_{\alpha_0}^\alpha) W_{\alpha_0}$. Repeating the arguments of part b) of the preceding theorem, we obtain that P_α form an $(n-1)$-dimensional simplicial spectrum, whose space contains the boundary of the basic neighborhood of the set Φ induced by W_{α_0}.

For $n = 0$ Theorem 3 is true; by induction it extends to the case of arbitrary n.

§ 3. The proof of Theorem 4 rests on some auxiliary considerations. Suppose two spectra are given: $S_X = \{X_\alpha, \wp_\alpha^{\alpha'}\}$ and $S_Y = \{Y_\beta, \wp_\beta^{\beta'}\}$, whose sets of indices we denote respectively by $A = \{\alpha\}$ and $B = \{\beta\}$, and whose limiting spaces by X and Y. Partially order the set C of all pairs $\gamma = (\alpha, \beta)$ by putting $(\alpha', \beta') > (\alpha, \beta)$ if simultaneously $\alpha' > \alpha$ and $\beta' > \beta$. To the index $\gamma = (\alpha, \beta)$ we put in correspondence the space $Z_\gamma = X_\alpha \times Y_\beta$, and for $\gamma' > \gamma$ define the projection $\wp_\gamma^{\gamma'}$ by the formula

$$\wp_\gamma^{\gamma'}(x_{\alpha'}, y_{\beta'}) = (\wp_\alpha^{\alpha'} x_{\alpha'}, \wp_\beta^{\beta'} y_{\beta'}).$$

Thus we obtain a spectrum $S_Z = \{z_\gamma, \wp_\gamma^{\gamma'}\}$, which we call the **product of the spectra** S_X and S_Y. It is easy to verify that the space Z of the spectrum S_Z is homeomorphic to the product $X \times Y$.

§ 4. **Proof of Theorem 4.** It is known that every connected finite-dimensional bicompact group is simply a compactum ((4), p. 107), i.e. is the space of a simplicial spectrum of the corresponding dimension ((5) Ch. IV, p. 183). From Mostert’ s results ((6)) it follows that the space of every bicompact group G decomposes into a topological product of the space of the connected component K of its identity and the space of the quotient group by this component. This quotient group is zero-dimensional, i.e. it has a zero-dimensional simplicial spectrum, and the spaces that constitute it consist of a finite number of points. Multiplying the simplicial spectra for K and G/K, we obtain a simplicial spectrum for G of the same dimension as for K, i.e.

$$\dim G = \operatorname{ind} G = \operatorname{Ind} G = \dim K,$$

which was to be proved.

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Note: Figure translations are in progress. See original paper for figures.

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