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Abstract

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A NOTE ON PETROVSKY' S CRITERION FOR THE UNIFORM WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS

I. G. Petrovsky's condition for the uniform well-posedness of the Cauchy problem for equations with two independent variables and constant coefficients of the form

$$Lu \equiv \frac{\partial^n u}{\partial t^n} + \sum_{k<n} A_{k,l} \frac{\partial^{k+l} u}{\partial t^k \partial x^l} = F \tag{1}$$

concerns the form of the operator L. This condition consists in the following (1). Consider the equation

$$\Delta(\lambda, \alpha) \equiv \lambda^n + \sum_{k<n} A_{kl} \lambda^k \alpha^l = 0. \tag{2}$$

In order that equation (1) admit in the infinite domain a solution satisfying the conditions

$$u\Big|_{t=0} = \frac{\partial u}{\partial t}\Big|_{t=0} = \cdots = \frac{\partial^{n-1} u}{\partial t^{n-1}}\Big|_{t=0} = 0, \tag{3}$$

depending continuously on the coefficients $A_{k,l}$ and on the right-hand side F, it is necessary and sufficient that, for purely imaginary values of α, all roots of equation (2) lie to the left of some straight line

$$\sigma > \sigma_0, \qquad \text{where } \lambda = \sigma + it. \tag{4}$$

Equations for which Petrovsky' s condition is satisfied will, for brevity, be called **Petrovsky equations**, and the operators on the left-hand side of such equations **Petrovsky operators**. In the present note we shall show that all Petrovsky operators can be represented in a certain canonical form. The proof is based on several elementary lemmas.

We shall call a Petrovsky operator of the form

$$Lu \equiv \left(\frac{\partial}{\partial t} - A\frac{\partial^m}{\partial x^m}\right)u. \tag{5}$$

an **elementary Petrovsky operator**.

Lemma 1. For elementary Petrovsky operators of odd order, the number A is always real.

Lemma 2. For elementary Petrovsky operators of even order $m = 2k$,

$$\left|\arg\left[(-1)^{k+1}A\right]\right| \leq \pi/2. \tag{6}$$

We shall call an elementary Petrovsky operator with even m **strictly parabolic** if inequality (6) is satisfied strictly.

Operators with odd m, or those for which, with even m, inequality (6) becomes an equality, will be called **semihyperbolic**.

Theorem 1. *A binomial differential operator of the form*

$$Lu \equiv \left(\frac{\partial^n}{\partial t^n} - B\frac{\partial^p}{\partial x^p}\right)u \tag{7}$$

will be a Petrovskii operator if and only if $n = 2$, p is even, $p = 2m$, and both factors

$$\frac{\partial}{\partial t} - \sqrt{B}\frac{\partial^m}{\partial x^m}, \qquad \frac{\partial}{\partial t} + \sqrt{B}\frac{\partial^m}{\partial x^m}, \tag{8}$$

into which this operator decomposes, are Petrovskii operators of semihyperbolic type.

For the proof it is enough to observe that from the equation

$$\lambda^n = B\alpha^p \tag{9}$$

it follows that

$$n\arg\lambda = \arg B + p\arg\alpha \tag{10}$$

or, for $\arg\alpha = \pi/2 + k\pi$,

$$\arg\lambda = \frac{\arg B}{n} + \frac{p\pi}{2n} + \frac{kp\pi}{n}. \tag{11}$$

If p/n is a fraction with irreducible denominator q greater than one, then there exist such k and s that

$$\frac{kp\pi}{n} = 2s\pi + \frac{\pi}{q}, \tag{12}$$

and hence, by varying k, one can force $\arg\lambda$ to take any value differing from the initial one by $j\pi/q$. Hence at least one of the values of λ will not satisfy Petrovskii' s condition. It follows from this that p/n is an integer.

But in this case equation (9) decomposes into factors:

$$\prod \left(\lambda - \sqrt[n]{B}\, e^{2k\pi i/n} \alpha^m\right) = 0. \tag{13}$$

In order that Petrovskii' s condition be satisfied, it is necessary and sufficient that all $\sqrt[n]{B}\, e^{2k\pi i/n}$ satisfy the conditions of Lemmas 1 and 2; this is possible only for $n = 2$. Finally, if one of the factors (8) were purely parabolic, then the other would not satisfy Petrovskii' s condition. The theorem is proved.

We proceed to the investigation of the Petrovskii operator in the general case. From the general theory of algebraic equations it is known that the values $\lambda(\alpha)$ of all roots of equation (2), in a neighborhood of the point at infinity $\lambda = \infty$, can be expanded into power series in decreasing powers of α, in a neighborhood of the point at infinity. These expansions will be

$$\lambda = A_0\alpha^m + A_1\alpha^{m-1/s} + \ldots + A_{ms} + \ldots \tag{14}$$

To find the principal terms we proceed by Newton' s method. To each monomial entering $\Delta(\lambda, \alpha)$, of the form $A_{k_s l_s}\lambda^{k_s}\alpha^{l_s}$, we put into correspondence the straight line

$$y_s(t) = k_s t + l_s. \tag{15}$$

In the first coordinate angle $t \geq 0,\ y \geq 0$ we construct the function

$$z(t) = \max_s y_s(t).$$

To each vertex of the broken line $z(t)$ there correspond numbers A_0 and m.

We shall call **senior terms of equation** (2) those terms for which the corresponding straight line (15) has at least one common point with the curve $z(t)$. The remaining terms will be called **junior**.

Theorem 2. *In the sequence of terms $A_k\alpha^{m-k/s}$ in formula (14) with $A_k \neq 0$, besides the hypoelliptic expressions satisfying the conditions of Lemmas 1 and*

2, there may also occur terms of four types: a) purely parabolic terms; b) terms with nonpositive powers of α; c) terms with fractional powers of α; d) terms not satisfying the Petrovskii condition.

If the first of these terms is a term of type a) or b), then the Petrovskii condition for this root will be satisfied; if it is of type c) or d), the Petrovskii condition will not be satisfied.

The proof is obvious.

Theorem 3. *Every Petrovskii operator, up to junior terms, can be represented in the form of a product of elementary Petrovskii operators:*

$$Lu \equiv \prod_{s=1}^{n} \left(\frac{\partial}{\partial t} - A_s \frac{\partial^{m_s}}{\partial x^{m_s}} \right) u + L_2 u. \tag{16}$$

Here $L_2 u$ represents the collection of terms of lower order, and all

$$\frac{\partial}{\partial t} - A_s \frac{\partial^{m_s}}{\partial x^{m_s}}$$

are elementary Petrovskii operators. The converse, obviously, is false. Not every operator of the form (15) will be a Petrovskii operator.

The proof of Theorem 3 follows from the fact that among the principal terms of the expansion of the roots λ of equation (2) there can be no terms which contain fractional powers of α or do not satisfy the Petrovskii condition.

In the case where, among the roots $\lambda(\alpha)$ of the determinant $\Delta(\lambda, \alpha)$ for some Petrovskii operator, there is such a root in whose expansion positive fractional powers of α participate:

$$\lambda_j = A_0 \alpha^m + A_1 \alpha^{m-1/s} + \dots + A_{m_s} + \cdots, \tag{17}$$

then, obviously, all the roots conjugate to it will also be roots, i.e. those obtained by substituting, instead of $\alpha^{1/s}$, the values $e^{2k\pi i/s} \alpha^{1/s}$.

The product

$$R(\lambda, \alpha) \equiv \prod_{k=0}^{s-1} \left(\lambda - A_0 \alpha^m - A_1 \alpha^{m-1/s} - \dots - A_{m_s} \right) \tag{18}$$

will, obviously, be a polynomial in λ and α, since it is a symmetric function of the roots of the equation $y^s = \alpha$. We shall call such polynomials $R(\lambda, \alpha)$ **canonical**, and the operator $R(\partial/\partial t, \partial/\partial x)$ a **canonical Petrovskii operator**. For $s = 1$ the canonical Petrovskii operator will have first degree with respect to $\partial/\partial t$.

Let L be some Petrovskii operator. Consider all roots $\lambda(\alpha)$ of the corresponding equation (17) and put each group of roots in correspondence with a canonical Petrovskii operator. Form the product of all such canonical operators

$$L_1 u \equiv \prod_{t=0}^{N} R_t\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u.$$

We shall now construct one more broken line in the y-t plane, smaller than (16), according to the following rule.

Arrange the exponents m_s of the principal terms in the expansions of the roots of the equation $\Delta(\lambda, \alpha) = 0$ in nondecreasing order:

$$m_1 \leqslant m_2 \leqslant \cdots \leqslant m_N$$

and associate with each m_s the number

$$n_s = (s-1)m_s + \sum_{j=s+1}^{n} m_s.$$

In our y, t plane construct a concave broken line joining the points $t = m_s,\ y = n_s$. We shall call the term $A_{kl}\partial^{k+l}u/\partial t^k \partial x^l$ an inessential term of the equation $Lu = f$ if the straight line $y = tk + l$ constructed for it has no common points with this broken line and lies everywhere below it.

Then the following theorem holds.

Theorem 4. The exterior Petrovsky operator Lu differs by inessential terms from the product of the canonical Petrovsky operators $Lu = L_1 u + L_2 u$. Conversely, every operator which, up to inessential terms, coincides with the product of canonical Petrovsky operators is itself a Petrovsky operator.

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References

1. I. G. Petrovsky, *Bull. Moscow State Univ.*, vol. 7, 23 (1938).

Note: Figure translations are in progress. See original paper for figures.