Mathematics

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Abstract

Full Text

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On Limit Points of the Set of Markov Numbers

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I. Let α be an arbitrary irrational number of the interval $(0, 1)$,

$$\alpha = [0;\ a_1, a_2, \dots, a_k, \dots] \tag{1}$$

its expansion into an arithmetic continued fraction.
Let $M(N)$ be the set of all numbers α for which

$$\varlimsup_{k\to\infty} a_k = N \qquad (N = 1, 2, \dots); \tag{2}$$

then

$$M = \sum_{N=1}^{\infty} M(N) \tag{3}$$

is the set of all irrational numbers of the interval $(0, 1)$ with a bounded sequence of partial quotients a_k.

Denote by $L(\alpha)$ the lower bound of the set of all real numbers $c > 0$ for which the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2} \tag{4}$$

has infinitely many solutions in integers $q > 0, p$.
As is known ((1), p. 29),

$$L(\alpha) = \frac{1}{\varlimsup_{k\to\infty}\left([a_k;\ a_{k+1}, \dots] + [0;\ a_{k-1}, a_{k-2}, \dots, a_1]\right)}. \tag{5}$$

For brevity, let us introduce the notation:

$$\lambda_k(\alpha) = [a_k;\ a_{k+1}, \dots] + [0;\ a_{k-1}, a_{k-2}, \dots, a_1], \tag{6}$$

$$\lambda(\alpha) = \varlimsup_{k\to\infty} \lambda_k(\alpha). \tag{7}$$

From (5), (6), and (7) it follows that

$$L(\alpha) = \frac{1}{\lambda(\alpha)}. \tag{8}$$

Let $M_L(N)$ be the set of values of $L(\alpha)$ as α ranges over the whole set $M(N)$; then

$$M_L = \sum_{N=1}^{\infty} M_L(N) \tag{9}$$

is the set of all values of the function $L(\alpha)$. The values of the function $L(\alpha)$ will be called by us **Markov numbers**, so that, in this terminology, M_L is the set of all Markov numbers.

Several facts are known about the set of Markov numbers (see (^{2}), pp. 121-132). The present paper is devoted to a further investigation of the structure of this set, more precisely, of its subsets $M_L(N)$.

Lemma. If $\{k_n\}$ is a sequence of indices of extreme positions of the number N in the expansion α, $\alpha \in M(N)$, then for $N = 2, 3$

$$\lambda(\alpha) = \varlimsup_{k\to\infty} \lambda_k(\alpha) = \varlimsup_{n\to\infty} \lambda_{k_n}(\alpha). \tag{10}$$

Theorem 1. The minimal point of the set $M_L(N)$ $(N \geqslant 2)$ is an accumulation point of this set.

Proof. As is known ((^{1}), p. 33), the set $M_L(N)$ is contained in the segment

$$[(N^2 + 4N)^{-1/2},\ (N^2 + 4)^{-1/2}],$$

and the endpoints of this segment belong to the set $M_L(N)$. Consequently, it is required to prove that the number $(N^2 + 4N)^{-1/2}$ is an accumulation point of $M_L(N)$.

1) $N \geqslant 4$. Consider the expression

$$V_{l,m}(x, y) = [N;\ (1, N)_l, N, x] + [0;\ (1, N)_m, N, y], \quad 1 \leqslant x, y \leqslant N + 1. \tag{11}$$

From (11) it follows that, uniformly with respect to x, y in the indicated domain, the equality holds

$$\lim_{l,m\to\infty} V_{l,m}(x, y) = N + 2[0;\ (1, N)_\infty] = (N^2 + 4N)^{-1/2}, \tag{12}$$

where

$$V_{l,m}(1,1) \leqslant V_{l,m}(x,y) \leqslant V_{l,m}(N+1,N+1) \leqslant (N^2+4N)^{-1/2}, \tag{13}$$

as follows directly from the properties of continued fractions.

Let $\varepsilon > 0$ be any prescribed arbitrarily small number; then, on the basis of (12) and (13), there exists $N_1 = N_1(\varepsilon)$ such that for $l, m > N_1$, for any pair x, y from the indicated domain, the inequality holds

$$(N^2+4N)^{-1/2} - \varepsilon < V_{l,m}(x,y) < (N^2+4N)^{-1/2}. \tag{14}$$

Next, let $\overline{\mathfrak{M}}(N-2)$ be the set of all numbers

$$\beta = [0;\ b_1, b_2, \dots, b_k, \dots],$$

defined by the condition

$$1 \leqslant b_k \leqslant N-2 \quad (k \geqslant 1). \tag{15}$$

$\overline{\mathfrak{M}}(N-2)$ has the cardinality of the continuum.

Let $l_0, m_0 > N_1$ be a fixed pair of natural numbers; then, by virtue of (14),

$$(N^2+4N)^{-1/2} - \varepsilon < V_{l_0,m_0}(x,y) < (N^2+4N)^{-1/2}. \tag{16}$$

Consider the finite ordered system of natural numbers

$$\{N,\ (N,1)_{l_0},\ N_1(1,N)_{m_0},\ N\} = T \tag{17}$$

and some infinite increasing sequence $\{t_n\}$ of natural numbers, and define the number α by the expansion

$$\begin{aligned}\alpha = [0;\ b_1, b_2, \dots, b_{t_1}, b_{t_1}, b_{t_1-1}, \dots, b_1, T, b_1; b_2, \dots, b_{t_2}, b_{t_2}, b_{t_2-1}, \dots, b_1, T, \dots]\\ = [0;\ a_1, a_2, \dots, a_k, \dots].\end{aligned} \tag{18}$$

$\alpha \in M(N)$, and the set of all such α has the cardinality of the continuum.

Further, by means of an argument based on the same idea as in (2), it is established that all these values $L(\alpha)$ are distinct and lie in the segment

$$\left[\frac{1}{\sqrt{N^2+4N}}, \frac{1}{\sqrt{N^2+4N-\varepsilon}}\right],$$

whence the validity of the theorem follows for $N \geqslant 4$.

2) In the case $N = 2$ or $N = 3$ the validity of the theorem is established by analogous arguments; only the form of the constructed numbers α is changed in the corresponding way, and the lemma is used essentially.

Theorem 2. *The maximal point* $(N^2 + 4)^{-1/2}$ *of the set* $M_L(N)$ $(N > 1)$ *is its isolated point.*

II. **Theorem 3.** *The maximal point of condensation of the set* $M_L(3)$ *is equal to*

$$\frac{22}{65 + 9\sqrt{3}}.$$

Proof. It is known ((1), p. 33) that to the right of the number

$$\frac{22}{65 + 9\sqrt{3}}$$

there is the unique number

$$\frac{1}{\sqrt{13}}$$

of the set $M_L(3)$. Therefore it is enough to establish that in an arbitrarily small neighborhood of the number

$$\frac{22}{65 + 9\sqrt{3}}$$

there is a continuum subset of the set $M_L(3)$.

Let $\{m_k\}$ and $\{n_k\}$ be two arbitrary sequences of natural numbers tending to ∞ as $k \to \infty$; then, for the number α defined by the expansion

$$\alpha = [0; (1,2)_{m_1}, 3, 3, (2,1)_{n_1}, (1,2)_{m_2}, 3, 3, (2,1)_{n_2}, (1,2)_{m_3}, 3, 3, (2,1)_{n_3}, ...]$$
$$= [0; a_1, a_2, ...], \qquad \alpha \in M(3), \tag{19}$$

on the basis of (6), (7), and (8), the equalities

$$\lambda(\alpha) = [3; (2,1)_\infty] + ([3; (2,1)_\infty])^{-1} = \frac{65 + 9\sqrt{3}}{22} = A, \tag{20}$$

$$L(\alpha) = \frac{1}{\lambda(\alpha)} = \frac{1}{A} = \frac{22}{65 + 9\sqrt{3}}. \tag{21}$$

hold.

Next, let

$$U_{l,m}(x, y) = [3; (2,1)_l, x] + [0; 3, (2,1)_m, y], \qquad 1 \le x, y \le 3. \tag{22}$$

From (20) and (22) it follows that in the rectangle $[1,3;\ 1,3]$ the equality

$$\lim_{l,m \to \infty} U_{l,m}(x, y) = A \tag{23}$$

holds uniformly with respect to x, y.

Let ε be an arbitrarily small positive number; then, by virtue of (23), for $l, m > N = N(\varepsilon)$ the inequalities

$$A - \varepsilon < U_{l,m}(x, y) < A + \varepsilon, \qquad 1 \leq x, y \leq 3. \tag{24}$$

will hold.

If $l_0 > N$ is any fixed natural number, then for all sufficiently large m_0 $(m_0 > N)$ the sum

$$\left[0; (2,1)_{l_0}, \frac{1}{\beta}\right] + \left[0; 3, (2,1)_{m_0}, \frac{1}{\beta}\right] = U_{l_0, m_0}\left(\frac{1}{\beta}, \frac{1}{\beta}\right) - 3 \tag{25}$$

is an increasing function of β, by virtue of the corresponding properties of continued fractions.

Supposing l_0 and m_0 to have been chosen in the manner indicated above, consider the system

$$\{(1,2)_{l_0},\ 3,\ 3(2,1)_{m_0}\} = T \tag{26}$$

and the set $\overline{\mathfrak{M}}(2)$ of all numbers β defined by the conditions

$$\beta = [0;\ b_1, b_2, \dots, b_k, \dots], \qquad 1 \leq b_k \leq 2; \tag{27}$$

$\overline{\mathfrak{M}}(2)$ is a set of the cardinality of the continuum.

Taking an arbitrary sequence $\{t_n\}$ of natural numbers tending to infinity as $n \to \infty$, to each β, $\beta \in \overline{\mathfrak{M}}(2)$, we assign a number α, $\alpha \in M(3)$, defined by the expansion

$$\begin{aligned} \alpha = [0;\ & b_1, b_2, \dots, b_{t_1}, b_{t_1}, b_{t_1 - 1}, \dots, b_1, T,\ b_1, b_2, \dots, b_{t_2}, b_{t_2}, b_{t_2 - 1}, \dots \\ & \dots, b_1, T, b_1, b_2, \dots] = [0;\ a_1, a_2, \dots]; \end{aligned} \tag{28}$$

the set $E = \{\alpha\}$ of all such α has the cardinality of the continuum.

By arguments analogous to those used in the preceding paper, it is established that $L(\alpha)$ for $\alpha \in E$ is represented as follows:

$$L(\alpha) = \begin{cases} \psi_1(\beta), & \text{for } \beta \in \overline{\mathfrak{M}}_1(2), \\ \psi_2(\beta), & \text{for } \beta \in \overline{\mathfrak{M}}_2(2), \end{cases} \tag{29}$$

where

$$\overline{\mathfrak{M}}_1(2) + \overline{\mathfrak{M}}_2(2) = \overline{\mathfrak{M}}(2),$$

and $\psi_1(\beta)$ and $\psi_2(\beta)$ are monotone functions of β; hence it follows that the set of all such values $L(\alpha)$ has the cardinality of the continuum. All these values are situated in an arbitrarily small neighborhood of the number

$$\frac{1}{A} = \frac{22}{65 + 9\sqrt{3}},$$

whence the theorem follows.

Theorem 4. The minimal point $(N^2 + 4N)^{-1/2}$ of the set $M_L(N)$ is a point of the set $M_L(N + 1)$ for $N \geq 3$.

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References

[1] J. F. Koksma, *Diophantische Approximationen*, Berlin, 1936. [2] A. G. Kogonia, DAN, 78, No. 4, 637 (1951).

Note: Figure translations are in progress. See original paper for figures.

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