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Abstract

Full Text

MATHEMATICS

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ON LIMIT LOGICS

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A natural generalization of k -valued logics P_k (1) is the infinitely valued logic P_{\aleph_0} , which is the set of all functions $\Phi(x_1, x_2, \dots, x_n)$ whose arguments are defined on the set $E^{\aleph_0} = \{0, 1, 2, \dots\}$ and such that $\Phi(a_1, a_2, \dots, a_n) \in E^{\aleph_0}$ whenever $a_i \in E^{\aleph_0}$ ($i = 1, 2, \dots, n$). It is clear that P_{\aleph_0} contains a continuum of distinct functions depending on the variables x_1, x_2, \dots . However, in applications, in particular when considering interpretations of deductive calculi constructed with the aid of a finite number of primitive connectives, one has to consider not the whole class P_{\aleph_0} , but its subsets consisting of a countable number of functions. In what follows we shall be interested in closed subclasses \mathfrak{F} of P_{\aleph_0} , i.e. such subsets of functions that are invariant under the operation of superposition. With respect to these subclasses we shall assume that each of them contains a countable number of functions.

Definition. A system of functions $\mathfrak{P} = \{f(x_1, \dots, x_n)\}$ is said to be **homomorphically mapped** into a system $\mathfrak{Q} = \{g(y_1, \dots, y_n)\}$ if there exists a one-to-one correspondence of the arguments $x_i \leftrightarrow y_i$, and to each function $f(x_1, \dots, x_n) \in \mathfrak{P}$ there uniquely corresponds a function $g(y_1, \dots, y_n) \in \mathfrak{Q}$ depending on the corresponding arguments, with the property that to every superposition of functions from \mathfrak{P} that belongs to \mathfrak{P} there corresponds an analogous superposition of the corresponding functions of the system \mathfrak{Q} , which also belongs to \mathfrak{Q} . If, under this mapping, the homomorphism holds in both directions, then the systems \mathfrak{P} and \mathfrak{Q} are said to be **isomorphic**.

Definition. A closed class $\mathfrak{F} \subset P_{\aleph_0}$ is called **limit** (a limit logic) if: 1) \mathfrak{F} consists of a countable number of functions; 2) \mathfrak{F} contains homomorphic preimages of the k -valued logics P_k ($k = 2, 3, \dots$).

A limit logic is a generalization of k -valued logics and, obviously, occupies an intermediate position between k -valued logics and countably valued logic. Here the question naturally arises: how many pairwise nonisomorphic limit logics exist? In the present note an exhaustive answer to the question posed is given. Namely, it is established that the maximal cardinality of a set of pairwise nonisomorphic limit logics is equal to the continuum.

Below we consider functions $f(x, y) \in P_{\aleph_0}$ having the so-called **box structure**. We say that a function $f_0(x, y)$ has box structure if there exists a partition of

the set E^{\aleph_0}

$$E^{\aleph_0} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots,$$

$$\mathcal{E}_0 = \{0\}, \quad \mathcal{E}_1 = \{1, \dots, k_1\}, \quad \mathcal{E}_2 = \{k_1 + 1, \dots, k_2\}, \dots, \quad 0 < k_1 < k_2 < \dots,$$

such that $f_0(\mathcal{E}_i, \mathcal{E}_i) \subset \mathcal{E}_i$ for every set \mathcal{E}_i containing more than one element; $f_0(\mathcal{E}_i, \mathcal{E}_i)$ is equal either to $\{0\}$ or to \mathcal{E}_i if \mathcal{E}_i contains one element and $i \geq 1$; and $f(x, y) = 0$ in the remaining cases (Table 1).

Let $S(x)$ be a function mapping E^{\aleph_0} onto E^{\aleph_0} . We shall call the func-

tion $S^{-i}(f_0(S(x), S(y)))$, where f_0 has a box structure, a function with a box structure.

Lemma 1. *A function $h(x_1, \dots, x_n)$ that is a superposition* of the function with box structure $f_0(x, y)$ and the constants $0, 1, \dots$ is either a constant, or a function which on each box $\underbrace{\mathfrak{E}_i \times \dots \times \mathfrak{E}_i}_{n \text{ times}}$ assumes either values from \mathfrak{E}_i or is identically equal to zero, while outside the boxes it always assumes the value 0; moreover, $h(x_1, \dots, x_n)$ essentially depends on all its variables x_1, \dots, x_n .*

Table 1

$y \setminus x$	0	1 ... k_1	$k_1 + 1$... k_2	$k_2 + 1$...
0	0			
1, ..., k_1		\mathfrak{E}_1	0	
$k_1 + 1$		0	\mathfrak{E}_2	
1, ..., k_2				
$k_2 + 1, \dots$...

We conduct the proof by induction on superposition. For the initial functions the assertion is obvious. Let $h(x_1, \dots, x_n) = f_0(h_1(x_{i_1}, \dots, x_{i_r}), h_2(x_{j_1}, \dots, x_{j_m}))$. The following cases are possible:

- $h_1(x_{i_1}, \dots, x_{i_r}) \equiv c_1, h_2(x_{j_1}, \dots, x_{j_m}) \equiv c_2$. Then $n = r = m = 0$ and $h(x_1, \dots, x_n) \equiv f_0(c_1, c_2) = c$;
- $h_1(x_{i_1}, \dots, x_{i_r}) \equiv c_1$, while $h_2(x_{j_1}, \dots, x_{j_m}) \not\equiv \text{const}$. It follows that $r = 0$ and $m = n$. Therefore $h_2(x_{j_1}, \dots, x_{j_m}) = h_2^1(x_1, \dots, x_n)$, where h_2^1 is the function obtained from h_2 by a permutation of variables. By the inductive hypothesis, the assertion of the lemma is true for $h_2^1(x_1, \dots, x_n)$. Suppose that on the box $\underbrace{\mathfrak{E}_i \times \dots \times \mathfrak{E}_i}_{n \text{ times}}$ the function h_2^1 assumes values from \mathfrak{E}_i . Then on this box $h(x_1, \dots, x_n)$ assumes values from the set $f_0(c_1, \mathfrak{E}_i)$,

where $f_0(c_1, \mathfrak{E}_i) = \{0\}$ when $c_1 = \bar{\mathfrak{E}}_i$, and $f_0(c_1, \mathfrak{E}_i) \subset \mathfrak{E}_i$ when $c_1 \in \mathfrak{E}_i$. If, however, on the box $\underbrace{\mathfrak{E}_i \times \dots \times \mathfrak{E}_i}_{n \text{ times}}$ $h_2^1(x_1, \dots, x_n) \equiv 0$, then on it $h(x_1, \dots, x_n) \equiv f_0(c, 0) = 0$, i.e. h is identically equal to zero. Obviously, outside the indicated boxes $h(x_1, \dots, x_n) = 0$.

- c) $h_1(x_{i_1}, \dots, x_{i_r}) \not\equiv \text{const}$, $h_2(x_{j_1}, \dots, x_{j_m}) \not\equiv \text{const}$. For the functions h_1 and h_2 , by assumption, the assertion of the lemma is true. Suppose that on the box $\underbrace{\mathfrak{E}_i \times \dots \times \mathfrak{E}_i}_{n \text{ times}}$ both functions assume values from \mathfrak{E}_i . Then on it $h(x_1, \dots, x_n)$ assumes values from $f_0(\mathfrak{E}_i, \mathfrak{E}_i) \subset \mathfrak{E}_i$. If at least one of the functions h_1 or h_2 on this box is identically equal to zero, then $h(x_1, \dots, x_n)$ will also, under these conditions, be identically equal to zero. Since every point lying outside the boxes $\underbrace{\mathfrak{E}_i \times \dots \times \mathfrak{E}_i}_{n \text{ times}}$, cov-

* Here we assume that, in forming superpositions, the introduction of inessential variables is not allowed.

is necessarily either outside the boxes $\underbrace{\mathcal{E}_i \times \dots \times \mathcal{E}_i}_{r \text{ times}}$ (with respect to the variables x_{i_1}, \dots, x_{i_r}), or outside the boxes $\underbrace{\mathcal{E}_i \times \dots \times \mathcal{E}_i}_{m \text{ times}}$ (with respect to the variables x_{j_1}, \dots, x_{j_m}), then at least one of the functions h_1 or h_2 is equal to zero at this point. Therefore, outside the boxes $\underbrace{\mathcal{E}_i \times \dots \times \mathcal{E}_i}_{n \text{ times}}$, $h(x_1, \dots, x_n) = 0$. Consequently, in all cases $h(x_1, \dots, x_n)$ has the required form. In passing we have also established that $h(x_1, \dots, x_n)$ depends essentially on all of its variables x_1, \dots, x_n . Thus, the lemma is proved. An analogous result holds if $f_0(x, y)$ is replaced by $S^{-1}(f_0(S(x), S(y)))$.

Let $h(x, y) \not\equiv \text{const}$ be a superposition of the function $f(x, y)$ with a box structure and of the constants $0, 1, \dots$. On the basis of the proved Lemma 1, this function on the box $\mathcal{E}_i \times \mathcal{E}_i$ either takes values from the set \mathcal{E}_i , or is identically equal to zero on it. In the latter case the box $\mathcal{E}_i \times \mathcal{E}_i$ is subdivided into one-cell boxes with zero values. Thus the function h will also have a box structure. To each box function let us assign a sequence consisting of nonnegative integers and the symbol ∞ : n_1, n_2, \dots , where n_i is the number of boxes of the i -th order (the corresponding set \mathcal{E}_i contains i elements). Obviously, $0 \leq n_i \leq \infty$. This sequence is constructed uniquely from the function $h(x, y)$. Further, assign to the function $h(x, y)$ a real number $\gamma = 0, \gamma_1 \gamma_2, \dots$, defined as follows: $\gamma_i = 0$ if $n_i \leq 1$;

Table 2

$y \backslash x$	s_i	$s_i + 1$	\dots	$s_i + k_i - 2$	$s_i + k_i - 1$
s_i	$s_i + 1$	$s_i + 2$	\dots	$s_i + k_i - 1$	s_i

$y \backslash x$	s_i	$s_i + 1$	\dots	$s_i + k_i - 2$	$s_i + k_i - 1$
$s_i + 1$	$s_i + 2$	$s_i + 2$	\dots	$s_i + k_i - 1$	s_i
\dots	\dots	\dots	\dots	\dots	\dots
$s_i + k_i - 2$	$s_i + k_i - 1$	$s_i + k_i - 1$	\dots	$s_i + k_i - 1$	s_i
$s_i + k_i - 1$	s_i	s_i	\dots	s_i	s_i

$\gamma_i = 1$ if $n_i > 1$. We see that to every function $h(x, y)$ with a box structure there corresponds a real number.

Lemma 2. Let $h(x, y) \not\equiv \text{const}$ be a superposition of the function with a box structure $f(x, y)$ and of the constants $0, 1, \dots$, and suppose $f(x, y)$ has at most one box of the 1st order ($n_1 = 1$). Let γ and α be the real numbers corresponding, respectively, to $h(x, y)$ and $f(x, y)$. Then $\gamma \geq \alpha$.

Lemma 3. If \mathfrak{P} and \mathfrak{D} are two isomorphic sets of functions, and all constants belong both to \mathfrak{P} and to \mathfrak{D} , then the corresponding functions $f(x, y)$ and $g(x, y)$ with box structures are assigned equal real numbers $\alpha = \beta$.

The isomorphism of the sets \mathfrak{P} and \mathfrak{D} means the existence of a one-to-one correspondence between the constants belonging to these sets. Since \mathfrak{P} and \mathfrak{D} contain all constants, it follows that the corresponding functions $f(x, y)$ and $g(x, y)$ have an equal number of boxes of each order.

Theorem. *The cardinality of a maximal subset of all limit logics in P_{\aleph_0} that are pairwise nonisomorphic is equal to the continuum.*

In one direction the assertion is obvious, since the cardinality of the set

$y \backslash x$	0	1	\dots	$k - 2$	$k - 1$	\dots
0	1	2	\dots	$k - 1$	0	\dots
1	2	2	\dots	$k - 1$	0	\dots
\dots	\dots	\dots	\dots	\dots	\dots	\dots
$k - 2$	$k - 1$	$k - 1$	\dots	$k - 1$	0	\dots
$k - 1$	0	0	\dots	0	0	\dots
\dots	\dots	\dots	\dots	\dots	\dots	\dots

Table 3

$y \backslash x$	0	1	2	3	4	5	6	7	8	9...
0	1	0								
1	0	0								
2			3	4	2	0	0	0	0	
3			4	4	2	0	0	0	0	
4			2	2	2	0	0	0	0	

$y \backslash x$	0	1	2	3	4	5	6	7	8	9...
5			0	0	0	6	7	8	5	
6			0	0	0	7	7	8	5	
7			0	0	0	8	8	8	5	
8			0	0	0	5	5	5	5	
9										...

of limit logics does not exceed the cardinality of all countable subsets of P_{\aleph_0} .

Now we shall show that the maximal cardinality of a subset of pairwise non-isomorphic limit logics is not less than the continuum. For this purpose we construct a continuous family of pairwise nonisomorphic limit logics. Let α ($0 \leq \alpha \leq 1/2$) be an arbitrary real number. Consider one of its expansions into an infinite binary fraction (taking $1/2 = 0.011\dots$): $\alpha = 0, \alpha_1 \alpha_2 \dots$. Under the imposed restrictions $\alpha_1 = 0$. Define a family of functions $f_\alpha(x, y)$ in the following way. The function $f_\alpha(x, y)$ has a box structure (n_1, n_2, \dots) , where $n_i = \alpha_i + 1$, and in the table there first occur n_1 boxes of order 1, then n_2 boxes of order 2, and so on. Let the i -th box $\mathcal{E}_i \times \mathcal{E}_i$ have order $k_i \geq 2$, and let $\mathcal{E}_i = \{s_i, s_i + 1, \dots, s_i + k_i - 1\}$; then on these boxes ($i = 2, 3, \dots$) $f_\alpha(x, y)$ has the form presented in Table 2. Outside these boxes (i.e. outside boxes of order $k_i \geq 2$) $f_\alpha(x, y) = 0$. We note that on $\mathcal{E}_i \times \mathcal{E}_i$, $f_\alpha(x, y) = s_i + \varphi_i(x, y)$, where $\varphi_i(x, y) = \max(x, y) + 1 \pmod{k_i}$ is Webb's function².

Denote by \mathfrak{T}_α the subclass of functions in P_{\aleph_0} generated by superpositions of the functions $f_\alpha(x, y)$ and the constants $0, 1, \dots$. It is easy to see that \mathfrak{T}_α is countable and contains homomorphic images of the k -valued logics P_k ($k = 2, 3, \dots$). Let $\alpha \neq \beta$. For definiteness let $\alpha > \beta$. Suppose that \mathfrak{T}_α is isomorphic to \mathfrak{T}_β . Then the function $f_\beta(x, y)$ from \mathfrak{T}_β would correspond to a function $h(x, y)$ from \mathfrak{T}_α . On the one hand, on the basis of Lemma 3, the real number γ corresponding to $h(x, y)$ must be equal to β , i.e. $\gamma = \beta$. On the other hand, since $h(x, y)$ cannot be a constant, by Lemma 2, $\gamma \geq \alpha$. Comparing these results, we arrive at a contradiction. Consequently, \mathfrak{T}_α is not isomorphic to \mathfrak{T}_β , and the theorem is completely proved.

In conclusion we give examples of two limit logics. Let the functions $\psi_k(x, y)$ and $\psi(x, y)$ be defined by Table 3. It is obvious that the class \mathfrak{T} , generated by superpositions of the functions $\psi_2(x, y), \psi_3(x, y), \dots$, and the class Q , generated by superpositions of the function $\psi(x, y)$, are limit logics; moreover, \mathfrak{T} has a countable basis, while Q has a basis consisting of one function. Further, \mathfrak{T} contains not only homomorphic images of P_k , but also isomorphic copies of P_k .

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¹ S. V. Yablonskii, DAN, **95**, No. 6 (1954).

² D. Webb, Proc. Nat. Acad. Sci. **21**, 252 (1935).

Note: Figure translations are in progress. See original paper for figures.

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