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G. I. KAC

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Abstract

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G. I. KAC

ON THE FUNCTIONAL CLOSEDNESS OF COMPLETELY REGULAR SPACES

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A completely regular space G is called **functionally closed** (a Q-space) if for every completely regular space S containing G as a dense subspace, there exists a continuous function defined on G (possibly unbounded) which cannot be continuously extended to S (1).

A completely regular space G is called a **complete topological space** if it is complete with respect to some uniform structure of its own.

As usual, by βG we shall denote the maximal bicompact extension of G (2). The following definition of a complete topological space is equivalent to the one given above.

A completely regular space G is called a **complete topological space** if, for every point $x_0 \in \beta G - G$, there is a continuous generalized metric $\rho(x, x')$ on G which cannot be extended to a continuous generalized metric on $x_0 \cup G^*$.

A. A. Kubenskii (4) studied the connection between functionally closed and complete topological spaces. He proved that *every normal complete topological space of attainable cardinality is functionally closed* **. In the present note this theorem is generalized to the case of completely regular spaces. More precisely, it is proved:

Theorem 1. *A complete topological space of attainable cardinality is functionally closed.*

The validity of Theorem 1 follows from the result of A. A. Kubenskii cited above and from the following theorem.

Theorem 2. *If discrete spaces of cardinality not exceeding m are functionally closed, then any complete topological spaces of cardinality m are functionally closed* ***.

Lemma 1. *In order that a completely regular space G be functionally closed, it is necessary and sufficient that for every point $x_0 \in \beta G - G$ there exist a continuous function defined on G which cannot be continuously extended to $x_0 \cup G$ ($x_0 \cup G$ is considered in the relative topology as a subspace of βG).*

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* The equivalence to the usual definition follows easily from Theorem 1 (^{3}). A **generalized metric** is a function $\rho(x, x')$ satisfying all the conditions valid for an ordinary metric, except for the condition $\rho(x, x') = 0 \to x = x'$. A generalized metric $\rho(x, x')$ is continuous if the set of all points such that $\rho(x, x') < \varepsilon$ is open ($\varepsilon > 0,\ x'$ fixed).

** That is, of cardinality less than the first inaccessible. A cardinal $p > c$ is inaccessible if it cannot be represented in the form

$$p = \sum_{\alpha \in A} 2^{m_\alpha},$$

where all m_α and the set A have cardinalities less than p.

*** The formulations of Theorems 1 and 2 were communicated to the author by Yu. M. Smirnov.

The necessity of the assertion is obvious. Let us prove sufficiency. Suppose that G is not a functionally closed space. Then there exists a completely regular space S containing G as a dense subspace, and moreover every continuous function $f(x)$ given on G can be extended to a continuous function $F(y)$ defined on S. By K denote the bicompact extension of the space S and, consequently, of the space G. According to a well-known theorem (^{2}), there exists a continuous mapping $y = \psi(x)$ of the bicompactum βG onto K, leaving invariant the points of G ($\psi(x) = x$ for $x \in G$). By x_0 denote the preimage of some point of $S - G$. Obviously, $x_0 \in \beta G - G$. The function $F(\psi(x))$ is continuous on $x_0 \cup G$ and is an extension of the function $f(x)$. Thus, any function continuous on G can be extended continuously to $x_0 \cup G$. This proves the sufficiency of the assertion of the lemma.

Lemma 2. Let the function $f(x)$ be continuous and defined on a completely regular space G. Put

$$f_N(x) = \begin{cases} f(x) & \text{for points } x \in G \text{ at which } |f(x)| \le N; \\ N & \text{""} \quad x \in G, \ \text{""} \quad f(x) > N; \\ -N & \text{""} \quad x \in G, \ \text{""} \quad f(x) < -N. \end{cases}$$

The function $f_N(x)$ is bounded and continuous on G, and therefore can be extended continuously to βG (^{2}). Denote the extended function by $F_N(x)$. The function $f(x)$ can be extended continuously to $x_0 \cup G$ ($x_0 \in \beta G$) if and only if there exists a (finite) limit

$$\lim_{N \to \infty} F_N(x_0).$$

$\lim_{N \to \infty} F_N(x_0)$ is the value of the extended function at the point x_0.

Suppose that the function $f(x)$ can be extended continuously to $x_0 \cup G$, and let $\Phi(x)$ be the extended function. Choose an arbitrary $N > |\Phi(x_0)|$ and take so small a neighborhood $U(x_0)$ of the point x_0 that at its points the inequality $N > |\Phi(x)|$ holds. Thus, $\Phi(x) = F_N(x)$ for $x \in U(x_0) \cap G$, and therefore $\Phi(x_0) = F_N(x_0)$. Since N is an arbitrary sufficiently large number, we have

$$\Phi(x_0) = \lim_{N\to\infty} F_N(x_0).$$

Now suppose that there exists a finite limit $\lim_{N\to\infty} F_N(x_0)$. We show that the function

$$\Phi(x) = \begin{cases} \lim\limits_{N\to\infty} F_N(x_0) & \text{when } x = x_0; \\ f(x) & \text{when } x \in G, \end{cases}$$

defined on $x_0 \cup G$, is a continuous extension of the function $f(x)$. It suffices to prove the continuity of the function $\Phi(x)$ at the point x_0. Choose arbitrary $\varepsilon > 0$ and then N so large that the inequalities

$$N - |\Phi(x_0)| > 2\varepsilon, \qquad |\Phi(x_0) - F_N(x_0)| < \varepsilon. \tag{1}$$

hold. Next choose so small a neighborhood $U(x_0)$ of the point x_0 that for $x \in U(x_0) \cap G$ the inequality

$$|F_N(x_0) - F_N(x)| = |F_N(x_0) - f_N(x)| < \varepsilon. \tag{2}$$

holds. From inequalities (1), (2) it follows that

$$|f_N(x)| < |F_N(x_0)| + \varepsilon < |\Phi(x_0)| + 2\varepsilon < N,$$

and therefore $f_N(x) = f(x)$ for all $x \in U(x_0) \cap G$. But then, from inequalities (1), (2), we have

$$|\Phi(x_0) - f(x)| = |\Phi(x_0) - \Phi(x)| < 2\varepsilon$$

for all $x \in U(x_0) \cap G$. The assertion follows.

From the lemma just proved, the validity of the following lemma follows easily.

Lemma 3. Let G and H be completely regular spaces; let $R(x)$ be a continuous mapping of βG onto βH, with $R(G) = H$. Let $x_0 \in \beta G - G$, $y_0 = R(x_0) \in \beta H - H$, and let $\varphi(y)$ be a continuous function on H, not extendable

cannot be extended continuously to $x_0 \cup G$.

Lemma 4. Let $y = r(x)$ be a continuous mapping of a completely regular space G onto a completely regular space H. The mapping $r(x)$ can be extended to a continuous mapping of the bicompactification βG onto βH.

Proof of Theorem 2. Let G be a complete topological space of cardinality m; let x_0 be an arbitrary point of $\beta G - G$. Below we shall prove the existence of a function continuous on G which cannot be extended continuously to $x_0 \cup G$. By Lemma 1 it follows from this that G is functionally closed.

In accordance with the definition of a complete topological space given above, there exists a continuous generalized metric $\rho(x_1, x_2)$ on G which cannot be extended to $x_0 \cup G$.

Divide all points of the space G into disjoint classes, assuming that two points x_1 and x_2 of G belong to one and the same class under the condition $\rho(x_1, x_2) = 0$ (and only in this case). It is not difficult to see that the values of the function $\rho(x_1, x_2)$ depend only on the classes y_1 and y_2 to which the points x_1 and x_2 belong. Therefore the equality $\rho^*(y_1, y_2) = \rho(x_1, x_2)$, where x_1 and x_2 are points of the classes y_1 and y_2, respectively, defines a function $\rho^*(y_1, y_2)$ on the set of classes H.

It is easy to verify that the function $\rho^*(y_1, y_2)$ defines a metric on the set H. Further, H is regarded as a metric space with metric $\rho^*(y_1, y_2)$.

Put $y = r(x)$, where y is the class to which the point $x \in G$ belongs; $y = r(x)$ is a single-valued mapping of G onto H. From the continuity of the generalized metric $\rho(x, x')$ follows the continuity of the mapping $y = r(x)$. Thus $y = r(x)$ is a continuous mapping of the completely regular space G onto the metric (and, consequently, completely regular) space H. In accordance with Lemma 4, extend the mapping $r(x)$ to a continuous mapping $y = R(x)$ of the bicompactification βG onto βH. The function $\tilde{\rho}(x_1, x_2) = \rho^*(R(x_1), R(x_2))$ is defined on the set $R^{-1}(H) \supseteq G$. It is not difficult to verify that the function $\tilde{\rho}(x_1, x_2)$ defines a continuous generalized metric on $R^{-1}(H)$. If $x_1, x_2 \in G$, then

$$\tilde{\rho}(x_1, x_2) = \rho^*(R(x_1), R(x_2)) = \rho^*(r(x_1), r(x_2)) = \rho^*(y_1, y_2) = \rho(x_1, x_2).$$

Thus, $\tilde{\rho}(x_1, x_2)$ is an extension of the generalized metric $\rho(x_1, x_2)$ to $R^{-1}(H)$. By hypothesis, the generalized metric $\rho(x_1, x_2)$ cannot be extended to $x_0 \cup G$, hence $x_0 \notin R^{-1}(H)$, and consequently $y_0 = r(x_0) \notin H$.

We now make use of a theorem of Katětov (5), from which it follows directly, under the hypotheses of Theorem 2, that metric spaces of cardinality not exceeding m are functionally closed. By construction, the cardinality of the metric space H does not exceed the cardinality of the space G, and consequently H is functionally closed. By Lemma 1 there exists a continuous function $\varphi(y)$, defined on H, which cannot be extended continuously to $y_0 \cup H$; but then, by Lemma 3, the function $\varphi(R(x))$, continuous on G, cannot be extended continuously to $x_0 \cup G$.

Theorem 2 is proved.

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Note: Figure translations are in progress. See original paper for figures.

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