ON THE AREA MEASURE UNDER A QUASICONFORMAL MAPPING

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Abstract

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It is known that under a q-quasiconformal mapping with small $1-1/q=\varepsilon$, the mapping is close to a conformal one, and, under the corresponding normalization, close to the identity mapping ([1,2]). However, up to now estimates of the change of area under such a mapping have been unknown. In the present note an estimate is given for the change of area of the form $|\Delta\sigma| \leq \sqrt{\varepsilon}+O(\varepsilon)$ for q-quasiconformal mappings. Along the way, the fact is once again proved that under quasiconformal mappings, sets of measure zero go into sets of measure zero ([3]).

Let the rectangle $K: 0<x<\lambda,\ 0<y<1/\lambda$ of the plane $z=x+iy$ be mapped q-quasiconformally onto the rectangle $0<u<\mu,\ 0<v<1/\mu$ of the plane $w=u+iv$, in such a way that the vertices go into the corresponding vertices. Let us compute $\operatorname{Im}\int_\Gamma w(z)\,dz$, where Γ is the boundary of the rectangle K:

$$\operatorname{Im}\int_\Gamma w(z)\,dz=\int_\Gamma u\,dy+v\,dx=\int_0^{1/\lambda}\mu\,dy-\int_0^{\lambda}\frac{1}{\mu}\,dx=\frac{\mu}{\lambda}-\frac{\lambda}{\mu}.$$

On the other hand:

$$\left|\int_\Gamma w(z)\,dz\right|\leq\iint_K|-(u_y+v_x)+i(u_x-v_y)|\,dx\,dy\leq\left(\sqrt{q}-\frac{1}{\sqrt{q}}\right)\iint_K\sqrt{\frac{D(u,v)}{D(x,y)}}\,dx\,dy.$$

Comparing with the preceding, we obtain

$$\left|\frac{\mu}{\lambda}-\frac{\lambda}{\mu}\right|\leq\left(\sqrt{q}-\frac{1}{\sqrt{q}}\right)\iint_K\sqrt{\frac{D(u,v)}{D(x,y)}}\,dx\,dy. \tag{1}$$

Now let, for definiteness, $\mu\geq\lambda$. Subject the plane w to a compression in the direction of the v-axis by $t\geq1$ times and to a stretching along the u-axis by the same factor (in the case $\mu<\lambda$ we interchange u and v). In this case there will be obtained a mapping of the rectangle K onto a rectangle with sides $t\mu$ and $1/t\mu$. The characteristic $p(z)$, obviously, will not exceed $t^2 q$, and the Jacobian will not change. Then, according to (1):

$$\left|\frac{t\mu}{\lambda} - \frac{\lambda}{t\mu}\right| \leq \left(t\sqrt{q} - \frac{1}{t\sqrt{q}}\right) \iint_K \sqrt{\frac{D(u,v)}{D(x,y)}}\, dx\, dy.$$

Dividing both sides of the obtained inequality by t, passing to the limit as $t \to \infty$, and taking into account that $\mu \geq \lambda$, we obtain

$$\frac{1}{\sqrt{q}} \leq \iint_K \sqrt{\frac{D(u,v)}{D(x,y)}}\, dx\, dy. \tag{2}$$

We divide the rectangle K into two subsets S_1 and S_2. Applying Bunyakovsky' s inequality to (2), we obtain

$$\frac{1}{\sqrt{q}} \leq \left[\operatorname{mes} S_1 \cdot \iint_{S_1} \frac{D(u,v)}{D(x,y)}\, dx\, dy\right]^{1/2} + \left[\operatorname{mes} S_2 \cdot \iint_{S_2} \frac{D(u,v)}{D(x,y)}\, dx\, dy\right]^{1/2}. \tag{3}$$

From the inequality obtained one can derive yet another proof of the fact that a mapping from the closure of the class of continuously differentiable q-quasiconformal mappings carries a set of measure zero into a set of measure zero.

To prove this, let us first note that inequality (3) remains valid, since it is known that Green' s formula is legitimately applicable in the case of a mapping belonging to the closure of the class of q-quasiconformal mappings.

Denote the images of the sets S_1 and S_2 by S_1' and S_2'. It is clear that

$$\operatorname{mes} S_2' \geq \iint_{S_2} \frac{D(u,v)}{D(x,y)}\, dx\, dy.$$

If $\operatorname{mes} S_1 = 0$, then from (3) it follows that

$$\operatorname{mes} S_1' = 1 - \operatorname{mes} S_2' \leq 1 - \iint_{S_2} \frac{D(u,v)}{D(x,y)}\, dx\, dy \leq 1 - \frac{1}{q}. \tag{4}$$

If we assume that $\operatorname{mes} S_1' > 0$, then, considering the mapping in a neighborhood of some point of density of S_1', it is possible, by means of auxiliary conformal transformations, to obtain a mapping carrying a set of measure zero into a set whose measure is arbitrarily close to one, which contradicts (4).

Denote mes S_1 by σ, and mes S_1' by $\sigma + \Delta\sigma$. Then (3) gives

$$\frac{1}{\sqrt{q}} \leq \sqrt{\sigma(\sigma+\Delta\sigma)} + \sqrt{(1-\sigma)(1-\sigma-\Delta\sigma)}. \tag{3'}$$

Solving the last inequality with respect to $\Delta\sigma$ and denoting $1 - 1/q = \varepsilon$, we obtain

$$|\Delta\sigma| \leq \varepsilon|1-2\sigma| + 2\sqrt{\sigma(1-\sigma)}\sqrt{\varepsilon+\varepsilon^2}, \tag{5}$$

or, for small ε,

$$|\Delta\sigma| \leq \sqrt{\varepsilon} + O(\varepsilon). \tag{5'}$$

Inequality (5′) is easily transferred to the case of a mapping of the disk onto itself with the normalization $w(0) = 0$.

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CITED LITERATURE

[1] M. A. Lavrent' ev, *Matem. sborn.*, **42**, issue 2 (1935).
[2] P. P. Belinskii, DAN, **91**, No. 2 (1953).
[3] I. P. Pesin, DAN, **102**, No. 2 (1955).

Note: Figure translations are in progress. See original paper for figures.