Reports of the Academy of Sciences of the USSR

G. I. GUREVICH

1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1958. Volume 120, No. 5

THEORY OF ELASTICITY

G. I. GUREVICH

ON THE DEPENDENCE BETWEEN THE TENSORS OF STRESSES AND STRAIN RATES IN THE GENERAL CASE OF LARGE AND SMALL DEFORMATIONS

(Presented by Academician A. P. Aleksandrov, 19 X 1957)

It was shown earlier ($^{3-6}$) that Maxwell' s law of deformation (if one takes into account the dependence of the relaxation time T on the stresses) is general for bodies in the condensed state*. At the same time it was expressed in a form to which it reduces in the case of deformations small in comparison with unity. Here we shall obtain it for deformations not restricted by the preceding condition. Let ξ_i denote the current coordinates of the points of the body

$$\xi_i = x_i + u_i(x_1, x_2, x_3, t) \quad (i = 1, 2, 3), \tag{1}$$

where $x_i = \xi_i(t = 0)$; u_i are the projections of the displacement of the point under consideration. We assume that for $t < 0$ the body is not loaded and has no residual stresses. For $t > 0$ we deform the body in an arbitrary way, but so that at each given instant its elastically changed shape differs very little from the equilibrium shape (corresponding to the absence of elastic deformations)**. Then a significant change in the shape of the body develops only at the expense of residual (irreversible) deformation.

At the instant t, mark a point of the body, regarding it as the vertex of one of the angles of an elementary rectangular parallelepiped with edges parallel to the axes of the Cartesian coordinate system $(O\overline{X}_1\overline{X}_2\overline{X}_3)$, coinciding in direction with the principal axes of the elastic deformation at the given instant at the point considered. Let its coordinates in the indicated system be $\overline{\xi}_i$ $(i = 1, 2, 3)$. The lengths λ_i of the edges of the parallelepiped are

$$\overline{\xi}_i' - \overline{\xi}_i = \lambda_i = \lambda_{pi} + \lambda_{yi} = \lambda_{pi}(1 + \lambda_{yi}/\lambda_{pi}) = \lambda_{pi}(1 + \overline{e}_{ii}) \tag{2}$$

($\overline{\xi}_i'$ is the coordinate of the other end of the edge, λ_{pi} its equilibrium length, λ_{yi} its elastic change for the given instant). In ([4]) it was obtained that

$$(d\lambda_{pi}/dt)/\lambda_{pi} = (\overline{e}_{ii} - \theta/3)/T = (\overline{\sigma}_{ii} - \sigma_{\text{cp}})/2\mu T, \quad (3)$$

where $d\lambda_{pi}$ is the increment of λ_{pi} during the elementary time interval dt, $\overline{e}_{ii}$ is the elastic deformation; $\overline{\sigma}_{ii}$ and σ_{cp} are the normal and mean stresses;

---

* As in ([4]), what is meant are practically homogeneous and isotropic bodies only with two components of deformation—elastic (Hookean) and residual (a laminar flow process). Moreover, hardening and other phenomena that can be described by taking into account the change, in the course of deformation, of structural parameters of the basic dependence are not considered here.

** Large rotations of elementary volumes of the body (for example, in bending thin plates) are treated in the theory of elasticity by special disciplines (see, for example, ([2])). Similarly, the residual deformation of bodies under conditions of large elastic change of their shape must be the subject of a special investigation.

$\theta = \sigma_{\text{cp}}/K$ is the relative change of volume; K and μ are the moduli of bulk expansion and shear; T is given by the expression

$$T = T_0 \exp\{(U_0 - m\sigma_{\text{cp}} - n|\overline{\sigma}_{ii} - \sigma_{\text{cp}}|_{\max})/k\vartheta\}, \quad (4)$$

where ϑ is the temperature; k is Boltzmann's constant; T_0, U_0, m, n are structural parameters of the material (here we regard them, as well as ϑ, as constant). Let us differentiate the relation, valid for each given instant (Hooke' s law),

$$\bar{e}_{ii} = \frac{\lambda_{yi}}{\lambda_{pi}} = \left(\bar{\sigma}_{ii} - \frac{3\nu}{1+\nu}\sigma_{\text{cp}}\right)/2\mu \quad (i = 1, 2, 3), \quad (5)$$

where ν is Poisson' s ratio, and add to (3) the quantity found, $\frac{d\lambda_{yi}}{dt}\frac{1}{\lambda_{pi}}$. Then, taking (2) into account,

$$\frac{1}{\lambda_i}\frac{d\lambda_i}{dt} = \frac{d}{dt}\left[\ln(1 + \bar{e}_{ii})\right] + \frac{\bar{\sigma}_{ii} - \sigma_{\text{cp}}}{2\mu T}. \quad (6)$$

On the basis of (2) and (1),

$$d\lambda_i = d\bar{\xi}_i' - d\bar{\xi}_i = \left(\bar{v}_i + \frac{\partial \bar{v}_i}{\partial \bar{\xi}_i}\lambda_i\right)dt - \bar{v}_i dt, \quad (7)$$

where $\bar{v}_i = d\bar{u}_i/dt = d\bar{\xi}_i/dt$ (we restrict ourselves to the first term of the Taylor expansion, since λ_i is arbitrarily small). Hence*

$$\frac{1}{\lambda_i}\frac{d\lambda_i}{dt} = \frac{\partial \bar{v}_i}{\partial \bar{\xi}_i}. \tag{8}$$

Substituting (8) into (6), we find, taking (5) into account (bearing in mind that always, up to fracture loads, $\bar{e}_{ii} \ll 1$),

$$\frac{\partial \bar{v}_i}{\partial \bar{\xi}_i} = \frac{\bar{\sigma}_{ii} - \sigma_{\text{cp}}}{2\mu T} + \frac{1}{2\mu}\frac{d}{dt}\left[\bar{\sigma}_{ii} - \frac{3\nu}{1+\nu}\sigma_{\text{cp}}\right] \quad (i = 1, 2, 3), \tag{9}$$

which may be written in tensor form as

$$2\mu\mathcal{E} = \frac{dG}{dt} + \frac{G}{T} - \left\{\frac{3\nu}{1+\nu}\frac{d\sigma_{\text{cp}}}{dt} + \frac{\sigma_{\text{cp}}}{T}\right\} I \tag{10}$$

($\mathcal{E}$ and G are the tensors of strain rates and stresses, I is the unit tensor), independent of the choice of coordinate system. We shall now use a fixed Cartesian coordinate system $(OX_1X_2X_3)$, common to all points at all instants of time. Then (10) can be written in the form of six (taking into account the symmetry of the tensors $\mathcal{E}$ and G) relations between the components of $\mathcal{E}$ and G in the system $(OX_1X_2X_3)$:

$$\mu T\left\{\frac{\partial}{\partial \xi_j}\left[\frac{du_i}{dt}\right] + \frac{\partial}{\partial \xi_i}\left[\frac{du_j}{dt}\right]\right\} = \sigma_{ij} - \delta_{ij}\sigma_{\text{cp}} + T\frac{d}{dt}\left[\sigma_{ij} - \delta_{ij}\frac{3\nu}{1+\nu}\sigma_{\text{cp}}\right], \tag{11}$$

---

* From (8) and (5) it follows that

$$\sum_{i=1}^{3}\frac{\partial \bar{v}_i}{\partial \bar{\xi}_i} = \sum_{i=1}^{3}\frac{1}{\lambda_i}\frac{d\lambda_i}{dt} = \frac{1}{\Omega}\frac{d\Omega}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{3}\ln(1+\bar{e}_{ii})\right] = \frac{1}{1+\theta}\frac{d\theta}{dt} \simeq \frac{d\theta}{dt}. \tag{*}$$

($\Omega = \lambda_1\lambda_2\lambda_3$ is the volume of the elementary parallelepiped in the deformed state.) Together with the condition of constancy of mass $d(\rho\Omega)/dt = 0$, where ρ is the density of the material in the deformed state, (*) gives the continuity equation in Eulerian form

$$\sum_{i=1}^{3}\frac{\partial v_i}{\partial \xi_i} = \frac{1}{1+\theta}\frac{d\theta}{dt} \simeq \frac{d\theta}{dt}. \tag{**}$$

where $\delta_{ij} = 1$ $(i = j)$ or $\delta_{ij} = 0$ $(i \neq j)$ (current coordinates and other quantities in this system are denoted without overbars). In the usual way we obtain three equations of motion

$$\frac{\partial \sigma_{i1}}{\partial \xi_1} + \frac{\partial \sigma_{i2}}{\partial \xi_2} + \frac{\partial \sigma_{i3}}{\partial \xi_3} = \rho_0 \left(\frac{d^2 u_i}{dt^2} - \delta_{3i} g \right) \qquad (i = 1, 2, 3) \tag{12}$$

(g is the acceleration of gravity, ρ_0 is the density of the material in the undeformed state). (11) and (12) constitute a complete system of 9 equations for determining the 9 unknowns u_i and σ_{ij}, for prescribed initial and boundary conditions*.

Passing from differentiation with respect to ξ_i to differentiation with respect to x_i, we obtain from (11) (taking into account that $\theta \ll 1$)

$$\mu T \sum_{k=1}^{3} \left[\frac{\partial^2 u_i}{\partial t\, \partial x_1} \alpha_{kj} + \frac{\partial^2 u_j}{\partial t\, \partial x_k} \alpha_{ki} \right] = \sigma_{ij} - \hat{\delta}_{ij} \sigma_{\text{cp}} + T \frac{\partial}{\partial t} \left[\sigma_{ij} - \hat{\delta}_{ij} \frac{3\nu}{1+\nu} \sigma_{\text{cp}} \right], \tag{13}$$

where α_{mn} are the cofactors of the determinant (m is the row number)**

$$D = \begin{vmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & 1 + \frac{\partial u_3}{\partial x_3} \end{vmatrix}. \tag{14}$$

Correspondingly, (12) are transformed into

$$\sum_{r=1}^{3} \left[\frac{\partial \sigma_{ir}}{\partial x_1} \alpha_{1r} + \frac{\partial \sigma_{ir}}{\partial x_2} \alpha_{2r} + \frac{\partial \sigma_{ir}}{\partial x_3} \alpha_{3r} \right] = \rho_0 \left(\frac{\partial^2 u_i}{\partial t^2} - \delta_{3i} g \right). \tag{15}$$

Under the condition $\partial u_i / \partial x_i \ll 1$, (13) pass into the ordinary three-dimensional Maxwell equations (which, for $T = \infty$, reduce to Hooke' s law), and (15) into the ordinary equations of motion***. Note that, in the particular case when $u_2 = u_3 = 0$, and u_1 depends only on x_2 and t (one-dimensional shear), (13) coincide in form with the equations of small deformations for any value of $\partial u_1 / \partial x_2$. This made it possible in ([5]) to apply the ordinary Maxwell equation to the investigation of the flow of a number of anomalous liquids. Taking into account here the dependence of T on σ led to satisfactory agreement with experiment.

It may seem that (11) do not describe the liquid state accurately, since for a constant viscosity coefficient $\eta = \mu T$ they do not coincide, in the general case, with the dependence from which (taking (12) into account) the Navier–Stokes equations follow. The latter differs from (11) in that in it, instead of

---

* If ρ_0 and the other parameters in (11) and (4) vary in the course of deformation, or if it is necessary to take account of their dependence on hydrostatic pressure (for example, in studying the behavior of a substance in the depths of the Earth), etc., then additional relations to the basic dependence will be required.

** $D = 1 + \theta \simeq 1$ is the equation of continuity in Lagrangian variables.

*** In (11), (12), (13), and (15), σ_{ij} are the normal and tangential components of the stress vectors on such three elementary areas which, at the given instant, pass through the point under consideration perpendicular to three corresponding axes of the system $(OX_1X_2X_3)$. If it is necessary to follow stresses referred all the time to the same areas, then through these stresses one must express the σ_{ij} entering (13) and (15). This can be done on the basis of the known relations between stresses on intersecting areas, as is done in works on the nonlinear theory of elasticity, for example in ([2]).

$$T\frac{d}{dt}\left[\sigma_{ij} - \delta_{ij}\frac{3\nu}{1+\nu}\sigma_{\text{av}}\right]$$

there appear the terms* $^2/_3\mu T\delta_{ij}\,\text{div}\,\mathbf{v}$, which can be neglected only if the stage of unsteady flow is not considered (or if the fluid is assumed ideal, when $T = 0$ and these terms are absent altogether). The reason for the difference is clear from the example given by Maxwell himself ([1]): to a one-dimensional loading of a body with a constant rate of deformation v_ε there corresponds, according to Maxwell' s equation, the stress $\tau = \eta v_\varepsilon(1 - e^{-\mu t/\eta})$, whence it follows that the Newtonian dependence $\tau = \eta v_\varepsilon$, for any value of the viscosity $\eta = \mu T$, is valid only for steady flows (when $e^{-\mu t/\eta}$ can be neglected in comparison with 1). However, it is precisely this dependence that is taken as the basis for the derivation of the Navier–Stokes equations, which are then applied as equations of the general case. As a result one obtains, for example, in the well-known problem of the unsteady flow of a fluid between a wall moving with constant velocity and a fixed wall, that at the moment when the motion begins the stress at the moving wall is infinite, although the fluid particles have not yet been displaced from the initial position. Equations (11) (with account of (12)), however, give the correct result (an increase of τ with time from zero to the steady value). Thus, in the case of the usual problems of hydromechanics of a viscous fluid (steady flow), Maxwell' s equations (with account of (12)) and Navier–Stokes coincide.** In unsteady flow, however, only solutions based on Maxwell' s equations have physical meaning.

From what has been considered in the present and preceding works it can be seen that the Maxwell law (with account of the dependence of T on σ_i) provides a general basis for approaching problems not only in the region of small deformations (plastic deformation, creep, propagation of elastic waves in solid and liquid bodies), but also in the region of large deformations (for example, stamping, forging, rolling—in technology; deformation of rocks—in geophysics; processes of steady and unsteady flow of Newtonian and anomalous fluids—in hydromechanics). At the same time, of course, the difficulties of solving the resulting nonlinear equations are evident, not only for the general case but also for particular cases allowing various simplifications. The development of the theory and technique of machine computation makes the prospect of overcoming these difficulties more realistic. For such computations, the possibility may prove especially useful of solving problems from different fields of investigation on the basis of a common initial dependence, applied in accordance with the specifics of individual problems.

The present work was preceded by an investigation of one-dimensional deformation carried out jointly with A. L. Rabinovich. The experience of that investigation has been substantially used in the present work. The author takes this opportunity to express his deep gratitude to A. L. Rabinovich.

Institute of Physics of the Earth named after O. Yu. Schmidt
Academy of Sciences of the USSR

Received
7 X 1957

References Cited

[1] J. Cl. Maxwell, Phil. Trans., London, **157**, 49 (1867).
[2] V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, L.–M., 1948.
[3] G. I. Gurvich, Tr. Geophys. Inst. AS USSR, No. 21 (148), 49 (1953).
[4] G. I. Gurvich, Tr. Geophys. Inst. AS USSR, No. 21 (148), 49 (1953).
[5] G. I. Gurvich, Tr. Geophys. Inst. AS USSR, No. 31 (158), 107 (1955).
[6] G. I. Gurvich, Tr. Geophys. Inst. AS USSR, No. 30 (157), 314 (1955).

---

* We assume, as in the usual courses of hydromechanics, that the so-called volume viscosity is equal to zero. It can be shown that if it is not equal to zero, the subsequent conclusions are not changed.

** In this case, taking account of the dependence of T on the stressed state makes it possible to include within the sphere of hydromechanics of a viscous incompressible fluid the flow processes not only of Newtonian, but also of anomalous fluids, as well as processes of development of large residual deformation of bodies solid in the everyday sense.

Note: Figure translations are in progress. See original paper for figures.