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1958

SovietRxiv

Abstract

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MATHEMATICS

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ON RATIONAL APPROXIMATIONS TO IRRATIONAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS

In the well-known master' s dissertation of A. A. Markov (the elder) (^{1}), if one speaks in language more modern for us (see (^{2})), the problem of two-dimensional extremal lattices Γ was solved to a considerable extent for the case when the excluded region γ is $|u \cdot v| < 1$ and the additional condition is that only one point of the lattice O should lie in γ, at its center. To such lattices correspond Markov extremal classes $\{M\}$ of indefinite binary quadratic forms. It turned out that the areas λ_M of the fundamental parallelograms of such lattices Γ that are less than 3 form an increasing sequence

$$\sqrt{5},\ \sqrt{8},\ \sqrt{\frac{221}{25}},\ \ldots,$$

converging to the number 3. The question of what the further "Markov spectrum," i.e. the whole set of values λ_M, is remained unclear up to the present.

The basic lemma from which Markov proceeded consists in the fact that the partial quotients of every extremal class $\{M\}$ form a two-sided infinite sequence

$$\ldots\alpha_{-3}\alpha_{-2}\alpha_{-1}\alpha_0\alpha_1\alpha_2\alpha_3\ldots \tag{1}$$

of bounded natural numbers, and, conversely, every completely arbitrary two-sided infinite sequence of bounded natural numbers is the sequence of partial quotients of some extremal class $\{M\}$, where the area λ_M of the fundamental parallelogram of its lattice Γ is the least upper bound of the numerical values of the sums:

$$(0, a_{n-1}a_{n-2}\ldots) + a_n + (0, a_{n+1}a_{n+2}\ldots) \tag{2}$$

(where $(0, a_{n+1}a_{n+2}\ldots)$ is the symbol of a continued fraction), composed of elements of the sequence (1), considered for all n.

Fig. 1

Figure 1: Fig. 1

On the other hand, since the time of Lagrange it has been known that any irrational number θ has infinitely many such irreducible rational approximations p/q for which

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\lambda q^2}, \tag{3}$$

if $\lambda = 1$. Hurwitz (^{3}) showed that here one may take $\lambda = \sqrt{5}$, but that λ can no longer be replaced by a number greater than $\sqrt{5}$, valid for all θ. Let $\{\theta\} = \{L\}$ be a Lagrange class of irrationalities, i.e. the set of so-called equivalent irrationalities, that is, those which, starting from some places, have the same expansion into a continued fraction, or, in other words, are expressed in terms of one another unimodularly, integrally, by fractional-linear transformations. All irrationalities split into such nonintersecting classes, and the least upper bound λ_θ

those values of λ for which inequality (3) still has infinitely many solutions, has a definite finite value if and only if the partial quotients of the expansion of θ are bounded; and this value is the same for all numbers θ of one and the same class $\{L\}$, and therefore it may also be denoted by λ_L.

In the paper (^{3}) Gurvich notes that "the investigation of the numbers λ_L leads to considerations analogous to those carried out in Markov' s paper (^{1})" and states without proof the following two assertions: 1) for every irrational θ not equivalent to

$$\frac{-1+\sqrt{5}}{2},$$

inequality (3) has infinitely many solutions already for $\lambda = \sqrt{8}$; 2) if $\sqrt{5} < \lambda < 3$, then only for a finite number of classes $\{L\}$ does inequality (3) not have infinitely many solutions. It seems to us that the question of the connection between the classes $\{L\}$ and $\{M\}$ has remained little clarified up to now.

Fig. 1

In the present note we solve two questions: 1) we investigate this connection, using for this purpose one argument close to one already used by Mahler and others for another purpose; 2) we note that from the theorem of M. Hall, Jr. (^{4}), on sums of two continued fractions with partial quotients not exceeding 4, it follows immediately that the Markov spectrum, at least beginning with

$$5 + \sqrt{2} = 6.41\ldots,$$

Fig. 2

Figure 2: Fig. 2

is continuous, i.e. constitutes a ray. One detail of this part of the spectrum was considered by P. G. Kogoniya (^{5}).

Fig. 2

§ 1. **On the connection between $\{L\}$ and $\{M\}$.** Construct the Klein polygons for the expansion of θ into a continued fraction and then stretch affinely the resulting lattice Γ_θ with coefficient λ_θ along the axis v. Then one obtains (Fig. 1) a lattice $\Gamma_{\theta\lambda_\theta}$ having the property that the boundary of the domain $|u \cdot v| < 1$ will be the first hyperbola lying on the u-axis; moreover the area of its fundamental parallelogram is equal to λ_θ. Let Γ_M be the extremal Markov lattice (Fig. 2); then one of its points lies in $\cdots$

at the origin O, no other lies in the domain $|u{\cdot}v| < 1$, and either there are points of it lying on the boundary of this domain, or there is a sequence of its points hyperbolically approaching this boundary more and more closely. In the first case we shall call the lattice Γ_M **attainable**, and in the second **unattainable**. Our aim is to find the relation between Fig. 1 and Fig. 2. The search for this relation is based on the following reasoning, proposed for this purpose by one of the authors of the present note (A. Vinogradov).

By virtue of the choice of λ_θ there is a sequence of points of $\Gamma_{\theta,\lambda_\theta}$ hyperbolically approaching the boundary γ, and it may be chosen among admissible points. Let A, A_1, A_2 be such points, and let $E, E_1, E_2, \ldots$ be those reduced frames whose first ends are these points. Consider the sequence of those equiaffine transformations of the plane under which E_1 is transformed into E, E_2 is transformed into E, and so on. Under these transformations: 1) the lattice $\Gamma_{\theta,\lambda_\theta}$ is each time carried into itself, since all the reduced frames are principal; 2) the points $A_1, A_2, \ldots$ are each time carried into the point A; 3) the origin O remains fixed, the pair of asymptotes uv is transformed into certain pairs $u_1v_1, u_2v_2, \ldots$, and the domain γ into certain domains $\gamma_1, \gamma_2, \ldots$, where γ_n is the set of points whose areas of the coordinate parallelograms with respect to the axes $u_{nv}n$ are less than 1. From the sequence of pairs $u_{nv}n$ one can always choose a subsequence $u^{(1)}v^{(1)}, u^{(2)}v^{(2)}, \ldots$ converging to some limiting pair of straight lines u^*v^*, which, as is easy to see, by boundedness of the partial quotients will be distinct (this can perhaps be done in various ways; then choose one of them), and let γ^* be the corresponding limiting domain γ of u^*v^*. Denote the frames corresponding to the chosen subsequence by $E^{(1)}, E^{(2)}, \ldots$, and their first ends by $A^{(1)}, A^{(2)}, \ldots$.

Theorem 1. *The lattice $\Gamma_{\theta,\lambda_\theta}$, with respect to the domain γ^*, is some attainable Markov lattice.*

Let $\Gamma_{\theta,\lambda_\theta}(1+\varepsilon)$ be the lattice obtained from $\Gamma_{\theta,\lambda_\theta}$ by an affine stretching from the v-axis with coefficient $1+\varepsilon$, where ε is a fixed positive number. $\Gamma_{\theta,\lambda_\theta}(1+\varepsilon)$

already has in γ only a finite number of points not lying on the u-axis, and on the v-axis it has no points distinct from the point O, since θ is irrational. Therefore there is a strip $|u| < \delta$ such that in its intersection with γ there are no points of $\Gamma_{\theta,\lambda_\theta}(1+\varepsilon)$, except the point O. Under the transformations considered, the second axis is contracted all the time; and since the transformations are equiaffine, the first is stretched; therefore the strip considered expands without bound from the second axis. Hence in $\gamma^{(n)}$, for large n, all points of $\Gamma_{\theta,\lambda_\theta}(1+\varepsilon)$ will lie sufficiently far from the origin O along the axis $u^{(n)}$. No point M of $\Gamma_{\theta,\lambda_\theta}(1+\varepsilon)$, except the point O, lies in γ^*, for otherwise it would lie in all $\gamma^{(n)}$ for sufficiently large n, which is impossible, since for large n the image of the strip $|u| < \delta$ will cover this point M. But there are also no points of $\Gamma_{\theta,\lambda_\theta}$ distinct from O lying in γ^*, since, if there were such a point, then for sufficiently small ε the corresponding point of $\Gamma_{\theta,\lambda_\theta}(1+\varepsilon)$ would also lie in γ^*. On the other hand, the point A, by convergence of the points $A^{(1)}, A^{(2)}, \ldots$ to the boundary γ, lies on the boundary γ^*. And since all the transformations considered are equiaffine, $\lambda_L = \lambda_M$.

We shall say that the Lagrange class $\{\theta\} = \{L\}$ **rests upon** the Markov class $\{M\}$ thus found.

From this proof the following consequences are obtained:

“insertions,” which may be quite arbitrary. Because of the arbitrariness of these insertions, on one and the same attainable class $\{M\}$ there rests, generally speaking, a continuum of classes $\{L\}$. At the same time, in each case $\lambda_L = \lambda_M$.

II. If $\lambda < 3$, then the sequence (1) is periodic (Markov), and from its properties (derived by Markov) it follows that the insertions are such that the classes $\{L\}$ and $\{M\}$ are connected one-to-one, namely: $\theta = a_0, a_1 a_2 \ldots a_k (-)(-)(-)\ldots$, where $(-)$ are the periods of the sequence (1) corresponding to the class $\{M\}$.

§ 2. **Theorem 2.** *The set of numbers* $\lambda_L = \lambda_M$*, at least starting with* $\lambda = 5+\sqrt{2}$*, is a ray.*

By Hall' s theorem, jr., the sums of two infinite continued fractions $(0, a_1 a_2 \ldots) + (0, b_1 b_2 \ldots)$, where a_i and b_j are arbitrary natural numbers $\leqslant 4$, run through all numbers of the interval from $2 \cdot (0, 4141 \ldots) = \sqrt{2} - 1$ to $2 \cdot (0, 1414 \ldots) = 4\sqrt{2} - 4$. This interval has length $d = 3\sqrt{2} - 3 > 1$. If the sequence (1) is taken in the form

$$\ldots b_3 b_2 b_1 \alpha_0 a_1 a_2 a_3 \ldots,$$

where $\alpha_0 \geqslant 6$, then its sum (2) for $n = 0$ is greater than α_0, and for $n \neq 0$ is less than α_0, i.e. its upper bound is attained at $n = 0$, and it is equal to $\alpha_0 + (0, a_1 a_2 \ldots) + (0, b_1 b_2 \ldots)$. Thus, starting with $6 + \sqrt{2} - 1 = 5 + \sqrt{2}$, an interval of length d of continuous spectrum is obtained; starting with $6 + \sqrt{2}$, the next such interval of length d, and so on. These intervals cover the ray extending to the right and beginning at $5+\sqrt{2}$. The determination of the least $\bar{\lambda}$ starting from which the Markov spectrum forms a ray, and the investigation of the piece of the Markov spectrum between $\lambda = 3$ and $\bar{\lambda}$, will be given in the next note.

Received
7 IX 1957

CITED LITERATURE

[1] A. A. Markov, *On binary quadratic forms of positive determinant*, Master' s dissertation, St. Petersburg, 1880. [2] B. N. Delone and D. K. Faddeev, Izv. AN SSSR, ser. matem., No. 6, 505 (1947). [3] A. Hurwitz, Math. Ann., 39 (1891). [4] M. Hall jr., Ann. Math., 48, No. 4 (1947). [5] P. G. Kogonia, DAN, 118, No. 4 (1958).

Note: Figure translations are in progress. See original paper for figures.

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