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Abstract

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MATHEMATICS

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RINGS OF LINEAR UNBOUNDED OPERATORS WITH A RESOLUTION OF THE IDENTITY AND THEIR REPRESENTATIONS

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In the present paper it is proved that the totality $\mathfrak{A}(P)$ of all linear operators in a Hilbert space $\mathfrak{H}$ which commute, in a definite sense, with a resolution of the identity P, forms a ring with respect to naturally defined addition and multiplication of operators. The ring $\mathfrak{A}(P)$ is closed with respect to "uniform" as well as "strong" passage to the limit (see below). If the resolution of the identity P has finite multiplicity, then every operator $A \in \mathfrak{A}(P)$ is isomorphic to the operator of multiplication by a functional matrix in a certain space $\mathcal{L}_\sigma^2$ of vector functions square-integrable with respect to a matrix distribution function σ. At the same time the concept of "spectrality" of an unbounded operator is generalized ([1−4]), and a canonical form of a spectral operator, analogous to the canonical form of a self-adjoint operator, is also constructed.

By a **resolution of the identity** of the Hilbert space $\mathfrak{H}$ we shall here mean a family $P = \{P(\Delta)\}$ of linear bounded operators $P(\Delta)$, $\Delta \in B$ (B is the Borel field of subsets of the complex plane Z), satisfying the conditions: (I) $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$; (II) $P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2)$, if $\Delta_1 \cap \Delta_2 = \Lambda$ (Λ is the empty set); (III) $P(\Lambda) = 0,\ P(Z) = I$ (I is the identity operator); (IV) $\|P(\Delta)\| \le K < \infty$; (V) $P(\Delta)$ is a countably additive function of Δ in the sense of strong convergence of operators. Obviously, $P(\Delta)$, $\Delta \in B$, is a projection (idempotent) operator. Since the condition $[P(\Delta)]^* = P(\Delta)$ is not assumed to be fulfilled, the projections, generally speaking, will not be orthogonal. However, by virtue of (IV), the "angle of projection" cannot become arbitrarily small.*

A class $C = \{\Delta\}$ of bounded sets $\Delta \in B$ will be called ***P*-admissible** if: (VI) C contains the union of any of its elements; (VII) C contains every Borel subset of any of its elements; (VIII) C contains some increasing sequence $\{\Delta_n\}$, $n = 1, 2, ...,$ such that

$$P\left(\bigcup_{n=1}^{\infty}\Delta_n\right) = I.$$

It is proved that the intersection of any finite number of P-admissible classes is also a P-admissible class.

A closed operator A from $\mathfrak{H}$ into $\mathfrak{H}$ with dense domain of definition $\mathfrak{D}(A)$ will be called **commuting** with $P = \{P(\Delta)\}$ (more precisely, C-commuting), if there exists such a P-admissible class C that: (IX) $P(\Delta)\mathfrak{H} \subset \mathfrak{D}(A)$ for every $\Delta \in C$; (X) $P(\Delta)\mathfrak{D}(A) \subset \mathfrak{D}(A)$ for every $\Delta \in B$; (XI) $P(\Delta)Ax = AP(\Delta)x$ for every $x \in \mathfrak{D}(A)$ and $\Delta \in B$**. Let

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* A more general definition of a resolution of the identity in a Banach space belongs to Dunford ([1]).

** The expediency of introducing the notions of a P-admissible class and a C-commuting operator is suggested by the considerations contained in ([2]).

A is an operator, C-permutable with $P(\Delta)$. Put $A(\Delta) = AP(\Delta)$, $\Delta \in C$. We shall call the operator function $A(\Delta)$ a C-basis of the operator A.

Lemma. 1) $A(\Delta)$, $\Delta \in C$, is a linear bounded operator defined on all of $\mathfrak{H}$.

2) The C-basis $A(\Delta)$ satisfies the condition

$$(\alpha) \quad P(\Delta_1)A(\Delta_2) = A(\Delta_2)P(\Delta_1) = A(\Delta_1 \cap \Delta_2), \qquad \Delta_1, \Delta_2 \in C.$$

3) If for every $\Delta \in C$ $A(\Delta)$ is a linear bounded operator defined on all of $\mathfrak{H}$, and condition (α) is fulfilled, then there exists an operator A, C-permutable with P, such that $AP(\Delta) = A(\Delta)$, $\Delta \in C$.

4) If C_1 and C_2 are two P-admissible classes and the operator function $A(\Delta)$, defined on $C_1 \cup C_2$, is a C_1-basis of the operator A_1 and a C_2-basis of the operator A_2, then $A_1 = A_2$. In particular, $\mathfrak{D}(A_1) = \mathfrak{D}(A_2)$.

5) For every operator A permutable with P there exists a P-admissible class C_A, containing any P-admissible class C satisfying the conditions in the definition of permutability (condition (IX)).

We shall denote the totality of all operators A permutable with P by $\mathfrak{A}(P)$. Let $A, B \in \mathfrak{A}(P)$. Then the operator functions $A(\Delta) + B(\Delta)$, and also $A(\Delta)B(\Delta)$, $\Delta \in C_A \cap C_B$, satisfy condition (α). Consequently, by virtue of the lemma, they uniquely determine operators permutable with P, for which they themselves serve as the corresponding bases. These operators are taken, by definition, as the sum $A + B$ and the product AB. It is easy to see that $\mathfrak{A}(P)$ forms a ring (noncommutative) with respect to addition and multiplication of operators, or, more precisely, a noncommutative algebra of infinite rank over the field of complex numbers. We note that if $A, A_1 \in \mathfrak{A}(P)$, $AA_1 = A_1A = I$, then $A_1 =$

A^{-1} in the sense of the usual definition of the inverse operator. In particular, $\mathfrak{D}(A_1) = \mathfrak{R}(A)$ and $\mathfrak{R}(A_1) = \mathfrak{D}(A)$ ($\mathfrak{D}(\ )$ is the domain of definition; $\mathfrak{R}(\ )$ is the range).

We shall say that a sequence $\{A_n\}$, $n = 1, 2, \dots$, of operators from $\mathfrak{A}(P)$ converges uniformly (strongly) to an operator $A \in \mathfrak{A}(P)$, if there exists a P-admissible class C such that $C \subset C_A \cap C_{A_1} \cap C_{A_2} \cap \cdots$ and for every $\Delta \in C$ the sequence of (bounded) operators $\{A_n(\Delta)\} = \{A_{nP}(\Delta)\}$ converges uniformly (strongly) to the operator $A(\Delta) = AP(\Delta)$. The uniform, and also the strong, completeness of the ring $\mathfrak{A}(P)$ is proved. Obviously, strong convergence is weaker than uniform convergence. However, if $A_n \to A$ strongly in the sense of $\mathfrak{A}(P)$, $x \in \bigcap_{n=1}^{\infty} \mathfrak{D}(A_n)$, and the sequence of vectors $\{A_{nx}\}$ converges in norm, then $x \in \mathfrak{D}(A)$ and $A_{nx} \to Ax$.

It is possible to construct analytic functions $F(A)$ of operators of the ring $\mathfrak{A}(P)$. For this it is enough to verify that the operator function $F(A(\Delta))$ satisfies condition (α) and to define $F(A)$ as the operator having $F(A(\Delta))$ as its basis: $F(A)P(\Delta) = F(A(\Delta))$. The correspondence $F(A) \leftrightarrow F(\xi)$ thus obtained ($F(\xi)$ is a function analytic in some neighborhood of the spectrum of A) has the character of an algebraic and, roughly speaking, topological isomorphism.

We shall call the resolution of the identity $P = \{P(\Delta)\}$ **of finite multiplicity** if there exists a subspace $\mathfrak{G} \subset \mathfrak{H}$ of finite dimension such that the linear span of vectors of the form $P(\Delta)x$, $x \in \mathfrak{G}$, $\Delta \in B$, is dense in $\mathfrak{H}$.

If P has finite multiplicity and for some operator $A \in \mathfrak{A}(P)$ there exists an operator A^{-1} having dense domain of definition, then also $A^{-1} \in \mathfrak{A}(P)$. Moreover, in the case of a resolution of the identity of finite multiplicity it is possible to construct the correspondence $F(A) \leftrightarrow F(\xi)$ without assuming that $F(\xi)$ is analytic on the spectrum of A. In this case it is enough that $F(\xi)$ be differentiable a definite, finite number of times. (In some cases it is enough that $F(\xi)$ be P-measurable and finite almost everywhere.) Finally, $A^* \in \mathfrak{A}(P)$, if $A \in \mathfrak{A}(P)$.

Let $\sigma(\Delta) = [\sigma_{ij}(\Delta)]$, $i, j = 1, \dots, r$, $\Delta \in B$, be a matrix distribution function, i.e. the matrix $\sigma(\Delta)$ is Hermitian, nonnegative, and is a countably additive function of Δ. Denote by $\mathcal{L}^2_\sigma$ the Hilbert space of all (classes, equivalent to one another, of) vector functions $f(\lambda) = [f_1(\lambda), \dots, f_r(\lambda)]$, $\lambda \in Z$, square-integrable with respect to σ. The scalar product in $\mathcal{L}^2_\sigma$ is given by the formula*

$$\langle f, f \rangle = \int_Z f(\lambda)\, \sigma(d\lambda)\, \overline{f(\lambda)} = \int_Z f(\lambda)\, \Sigma(\lambda)\, \overline{f(\lambda)}\, \sigma_0(d\lambda).$$

Here we use the representation $\sigma(\Delta) = \int_\Delta \Sigma(\lambda)\, \sigma_0(d\lambda)$, where $\sigma_0(\Delta)$ is a numerical distribution function, for example the trace of $\sigma(\Delta)$. The Radon–Nikodym derivative $\Sigma(\lambda) = \sigma(d\lambda)/\sigma_0(d\lambda)$ is a nonnegative matrix for almost all λ (with respect to σ). We shall say that a functional matrix $\Gamma(\lambda) = [\Gamma_{ij}(\lambda)]$, $i, j = 1, \dots, r$, belongs to the class $\mathcal{G}_\sigma$, if the elements $\Gamma_{ij}(\lambda)$ are σ-measurable, almost everywhere finite functions of the complex variable $\lambda \in Z$. For $\Gamma(\lambda) \in \mathcal{G}_\sigma$, by $\mathfrak{D}(\Gamma)$

we denote the set of all those $f(\lambda) \in \mathcal{L}^2_\sigma$ for which also $f(\lambda)\Gamma(\lambda) \in \mathcal{L}^2_\sigma$. For $f \in \mathfrak{D}(\Gamma)$ put $(\Gamma f)(\lambda) = f(\lambda)\Gamma(\lambda)$. The operator Γ will be called the operator of multiplication by the functional matrix $\Gamma(\lambda)$. The operator Γ may be unbounded even when all elements $\Gamma_{ij}(\lambda)$ of the matrix $\Gamma(\lambda)$ are bounded. The formula holds

$$|\Gamma| = \operatorname{vrai\,max} |\Gamma(\lambda)|,$$

where

$$|\Gamma(\lambda)| = \sup_\alpha \left(\alpha\Gamma(\lambda)\Sigma(\lambda)\overline{\alpha\Gamma(\lambda)}\right)^{1/2} \left(\alpha\Sigma(\lambda)\overline{\alpha}\right)^{-1/2}.$$

For any matrix $\Gamma(\lambda) \in \mathcal{G}_\sigma$, the manifold $\mathfrak{D}(\Gamma)$ is dense in $\mathcal{L}^2_\sigma$, and the operator Γ is a closed operator. The adjoint operator Γ^* is the operator of multiplication by any matrix $\Gamma^*(\lambda) \in \mathcal{G}_\sigma$ satisfying the relation

$$\Gamma(\lambda)\Sigma(\lambda) = \Sigma(\lambda)\overline{\Gamma^*(\lambda)}$$

almost everywhere.

Put $\Pi(\Delta; \lambda) = \chi_\Delta(\lambda)E$, where E is the identity matrix and $\chi_\Delta(\lambda)$ is the characteristic function of the Borel set $\Delta \subset Z$. Let $\Pi(\Delta)$ be the operator of multiplication by the functional matrix $\Pi(\Delta; \lambda)$. The family $\Pi = \{\Pi(\Delta)\}$, $\Delta \in B$, is a resolution of the identity for the normal operator Λ of multiplication by the functional matrix λE.

Theorem 1. *For every resolution of the identity P of finite multiplicity in a Hilbert space $\mathfrak{H}$, there exists a space $\mathcal{L}^2_\sigma$ and a linear one-to-one and bicontinuous mapping M of the space $\mathfrak{H}$ onto $\mathcal{L}^2_\sigma$ such that $MP(\Delta)M^{-1} = \Pi(\Delta)$ for all $\Delta \in B$.*

Theorem 2. *In order that an operator T from $\mathcal{L}^2_\sigma$ into $\mathcal{L}^2_\sigma$ belong to the ring $\mathfrak{A}(\Pi)$, it is necessary and sufficient that T be the operator of multiplication by some functional matrix $\Gamma(\lambda)$ of class $\mathcal{G}_\sigma$.*

Finally, let us consider the so-called spectral operators. In the present paper an operator A from $\mathfrak{H}$ into $\mathfrak{H}$ is called a **spectral operator with resolution of the identity** P if: a) $A \in \mathfrak{A}(P)$; b) the spectrum of the operator induced by A on the subspace $P(\Delta)\mathfrak{H}$ is contained in the closure of the set Δ for all $\Delta \in C_A$. This definition is a generalization of the corresponding definition belonging to Bade ([2]).

The following analogue of Dunford' s theorem is valid:

Theorem 3. *In order that an operator $A \in \mathfrak{A}(P)$ be a spectral operator with resolution of the identity P, it is necessary and sufficient that*

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* For an arbitrary rectangular matrix $\Gamma = [\Gamma_{ij}]$, $i = 1, \dots, r$; $j = 1, \dots, s$, by $\overline{\Gamma}$ is denoted the Hermitian conjugate matrix. For example, if $\alpha = [\alpha_1, \dots, \alpha_r]$, $\beta = [\beta_1, \dots, \beta_r]$, then

$$\alpha\overline{\beta} = \alpha_1\overline{\beta}_1 + \dots + \alpha_r\overline{\beta}_r.$$

root part A, i.e. the operator $N = A - \int_Z \lambda P(d\lambda)^*$, on every subspace $P(\Delta)\mathfrak{H}$, $\Delta \in C_A$, induces a generalized nilpotent operator (in the sense of I. M. Gelfand).

The **multiplicity of the spectrum** of a spectral operator A with a resolution of the identity P will be called the multiplicity of P. The root part N of a spectral operator A with a spectrum of finite multiplicity may also be an unbounded operator. However, it is always a nilpotent operator: $N^s = 0$, where the nonnegative integer s does not exceed the multiplicity of the spectrum of A. Indeed, if M is the linear homeomorphism described in Theorem 1 and A is a spectral operator with a finite-multiplicity resolution of the identity P, then $MAM^{-1} = \Gamma$, where Γ is the operator of multiplication by a functional matrix of the form $\lambda E + N(\lambda)$; $N(\lambda)$ is, for almost all λ (with respect to σ), a nilpotent matrix.

In Theorem 1 the operator M is an isomorphism of the spaces $\mathfrak{H}$ and $\mathcal{L}^2_\sigma$. Let us note that the metric in the space $\mathcal{L}^2_\sigma$ can be changed so that the corresponding isomorphism, under which the operators $P(\Delta)$ pass into the operators $\Pi(\Delta)$ and, in general, the operators of the ring $\mathfrak{A}(P)$ into the operators of the ring $\mathfrak{A}(\Pi)$, will be a unitary operator.

In addition, let us note that, using the known results of the theory of self-adjoint operators, it is not difficult to prove the existence of a complete system of generalized eigenfunctions and associated functions for every spectral operator.

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* The integral $\int_Z \lambda P(d\lambda)$ exists in the sense of uniform convergence in $\mathfrak{A}(P)$. The difference $A - \int \lambda P(d\lambda)$ is understood in the sense of the definitions of arithmetic operations in $\mathfrak{A}(P)$.

Note: Figure translations are in progress. See original paper for figures.