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Abstract

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INVARIANT CONSTRUCTIONS ON AN m-DIMENSIONAL SURFACE IN AN n-DIMENSIONAL AFFINE SPACE

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With the current point Λ of the surface there is associated an arbitrary frame $\vec{\Lambda}, E_\alpha$ $(\alpha, \beta = 1, 2, \dots, n)$. The surface is given by the equation $d\vec{\Lambda} = \omega^a \Lambda^\alpha_a E_\alpha$ $(a, b, c, d, e, f = 1, 2, \dots, m)$, where ω^α are Pfaff forms determining the group of analytic transformations of the parametrization. Prolongations of this equation lead to a sequence of fields of fundamental objects of the surface. For every w the fundamental object $\{\Lambda\}_w$ of order w consists of the components $\Lambda^\alpha_a, \Lambda^{\alpha_1}_{a^1 a^2}, \dots, \Lambda^\alpha_{a^1,\dots,a^w}$, symmetric in the lower indices. It is known (1) that by a field of a fundamental object of sufficiently high order one may cover a field with any generating object. The aim of the paper is the construction and study of such coverings. The work is carried out by the invariant-group method of G. F. Laptev (1).

In the constructions the following geometric objects with constant components are used: $\mathcal{B}^{b^1\dots b^r}_{a^1\dots a^r} \equiv \delta^{b^1}_{(a^1} \cdots \delta^{b^r}_{a^r)}$, $\mathcal{B}^{b^1\dots b^s}_{a^1\dots a^r} \equiv 0$, $r \geqslant 1$, $s \neq r$; $\mathcal{T}^{b^1\dots b^s, c^1\dots c^{r+1-s}}_{a^1\dots a^r, b} \equiv \mathcal{B}^{b^1\dots b^s}_{(a\dots a^s} \mathcal{B}^{c^1\dots c^{r+1-s}}_{a^{s+1}\dots a^r)b}$; $\mathcal{E}_{\alpha_1\alpha_2\dots\alpha_n}$ is the unit n-vector; $\mathcal{E}_{a^1a^2\dots a^m}$ is the unit m-vector; $\mathcal{E}_{a^1_1\dots a^r_1,\ a^1_2\dots a^r_2, \dots,\ a^1_{m_r}\dots a^r_{m_r}}$ is a relative tensor, symmetric in the indices belonging to one series and skew-symmetric with respect to the series, where

$$m_r \equiv \frac{(m+r-1)!}{r!(m-1)!}.$$

1. Relative tensors are constructed

$$H_{\alpha_1\dots\alpha_q} = \Lambda_1^{\beta^1_1\dots\beta^1_{m_1}} \cdots \Lambda_{p-1}^{\beta^{p-1}_1\dots\beta^{p-1}_{m_{p-1}}} \mathcal{E}_{\beta^1_1\dots\beta^1_{m_1}\dots\beta^{p-1}_1\dots\beta^{p-1}_{m_{p-1}}\alpha_1\dots\alpha_q},$$

$$K_{a^1_1\dots a^p_1,\dots,a^1_q\dots a^p_q} = \Lambda^{\alpha_1}_{a^1_1\dots a^p_1} \cdots \Lambda^{\alpha_q}_{a^1_q\dots a^p_q} H_{\alpha_1\dots\alpha_q},$$

where

$$\Lambda_r^{\beta_1\dots\beta_{m_r}} \equiv \Lambda^{\beta_1}_{a^1_1\dots a^r_1} \cdots \Lambda^{\beta_{m_r}}_{a^1_{m_r}\dots a^r_{m_r}} \mathcal{E}^{a^1_1\dots a^r_1,\dots,a^1_{m_r}\dots a^r_{m_r}},$$

$$q = n - (m_1 + m_2 + \dots + m_{p-1})$$

and p is the number such that

$$m_1 + \dots + m_{p-1} < n \le m_1 + \dots + m_p.$$

The vanishing of the first (second) of these tensors is equivalent to a decrease in the dimension of the osculating plane of order $p-1$ (p).

2. A nonzero relative invariant cannot be covered by the object $\{\Lambda\}_{p-1}$, and therefore the problem of covering it by the object $\{\Lambda_p\}$ is of interest. In the case when q is either equal to 2, or is a divisor of m,

is nonzero, the relative invariant

$$K = L_{a^1_{11} \dots a^{p+p}_{11}, \dots, a^1_{q1} \dots a^{p+p}_{q1}} \cdots L_{a^1_{1m} \dots a^{p+p}_{1m}, \dots, a^1_{qm} \dots a^{p+p}_{qm}} \times \mathcal{E}^{a_{111}{}^1, \dots, a^1_{1i\ell}} \dots \mathcal{E}^{a_{q1}{}^{p+p}, \dots, a^{p+p}_{qm}},$$

where

$$L_{a^1_1 \dots a^{p+p}_1, \dots, a^1_q \dots a^{p+p}_q} = \mathfrak{B}^{b_1{}^1 \dots b^p_1 c^1_1 \dots c^p_1}_{a^1_1 \dots a^{p+p}_1} \cdots \mathfrak{B}^{b_q{}^1 \dots b^p_q c^1_q \dots c^p_q}_{a^1_q \dots a^{p+p}_q} K^{c_1{}^1 \dots c^p_1, \dots, c^1_q \dots c^p_q}_{b^1_1 \dots b^p_1, \dots, b^1_q \dots b^p_q}.$$

In the case when $x(x = m_p - q)$ is a divisor of m, the relative invariant

$$K^* = L_{a^1_{11} \dots a^{p+p}_{11}, \dots, a^1_{x1} \dots a^{p+p}_{x1}} \cdots L_{a^1_{1m} \dots a^{p+p}_{1m}, \dots, a^1_{xm} \dots a^{p+p}_{xm}} \times \mathcal{E}_{a^1_{11} \dots a^1_{1m}} \cdots \mathcal{E}_{a^{p+p}_{x1} \dots a^{p+p}_{xm}},$$

where

$$L^{a^1 \dots a^{p+p}, \dots, a^1 \dots a^{p+p}}_{1^1 \dots 1^{p+p}, \dots, x^1 \dots x^{p+p}} = \mathfrak{B}^{a_1{}^1 \dots a^{p+p}_1}_{b^1_1 \dots b^p_1 c^1_1 \dots c^p_1} \cdots \mathfrak{B}^{a_x{}^1 \dots a^{p+p}_x}_{b^1_x \dots b^p_x c^1_x \dots c^p_x} \times K^{b_1{}^1 \dots b^p_1, \dots, b^1_x \dots b^p_x} K^{c_1{}^1 \dots c^p_1, \dots, c^1_x \dots c^p_x},$$

$$K^{b_1{}^1 \dots b^p_1, \dots, b^1_x \dots b^p_x} = \mathcal{E}^{b_1{}^1 \dots b^p_1, \dots, b^1_x \dots b^p_x, c^1_1 \dots c^p_1, \dots, c^1_q \dots c^p_q} K_{c^1_1 \dots c^p_1, \dots, c^1_q \dots c^p_q}.$$

In the case when $q = m_p$, the relative invariant

$$K^{**} = K_{a^1_1 \dots a^p_1, \dots, a^1_q \dots a^p_q} \mathcal{E}^{a_1{}^1 \dots a^p_1, \dots, a^1_q \dots a^p_q}.$$

The relative invariant K (similarly K^*, K^{**}) can be represented in the following forms:

$$K = \frac{1}{m_1} \Lambda^a_\alpha V^\alpha_a = \frac{1}{m_2} \Lambda^{a_1 a_2}_\alpha V^{a_1 a_2}_a = \dots = \frac{1}{m_{p-1}} \Lambda^\alpha_{a_1 \dots a_{p-1}} V^{a_1 \dots a_{p-1}}_\alpha = \frac{1}{q} \Lambda^\alpha_{a_1 \dots a_p} V^{a_1 \dots a_p}_\alpha.$$

Here $V^{a_1 \dots a_r}_\alpha$ are certain polynomials in the components of the object $\{\Lambda\}_p$. Their aggregate forms a geometric object. A sequence of geometric objects is also formed by the quantities

$$M^{b_1\dots b_s}_{a_1\dots a_r} = \frac{1}{K}\Lambda^{\alpha}_{a_1\dots a_r} V^{b_1\dots b_s}_{\alpha}.$$

For example, $M^{b_1\dots b_p}_{a_1\dots a_p}$ is a tensor.

3. The basis of the subsequent constructions is a sequence of normal objects, which correspond to the objects of connections of higher degrees of V. Hlavatý ([2]). The normal object $\{n\}_\omega$ consists of components $n^a_{a_1 a_2}$, $n^a_{a_1 a_2 a_3}$, ..., $n^a_{a_1\dots a\omega}$, symmetric in the lower indices. For $\{n\}_2$ the formula has been obtained

$$n^a_{a_1 a_2} = -W^{a,\,b_1 b_2}_{a_1 a_2,\,b} M^{bc_1\dots c^{p-1}}_{b_1 b_2 c_1\dots c^{p-1}},$$

in which $W^{a,\,b_1 b_2}_{a_1 a_2,\,b}$ are quantities uniquely determined by the linear system

$$\mathfrak{T}^{a_1 a_2,\,c_1\dots c^p}_{d^1 d^2 e^1\dots e^{p-1},\,a} M^{d^1 e^1\dots e^{p-1}}_{c_1\dots c^p} W^{a,\,b^1 b^2}_{a^1 a^2,\,b} = \mathfrak{B}^{b^1 b^2}_{d^1 d^2}\mathfrak{B}^d_b.$$

(The nondegeneracy of this system has been verified in the case when q is either equal to m_p, or is a divisor of m.) To cover the following normal objects, a group of formulas recurrent with respect to the index r (for arbitrary s) is given:

$$\mathfrak{K}^{a,b^1\dots b^s}_{a^1\dots a^s,b} = -\mathfrak{B}^{b^1\dots b^s}_{a^1\dots a^s}\mathfrak{B}^a_b, \qquad \mathfrak{K}^{a,b^1\dots b^s}_{a^1\dots a^{s+r},b} = \mathfrak{B}^{b^1\dots b^s}_{a^1\dots a^{s+r},b}\,\mathfrak{a}^{e^1\dots e^{r+1}}_{e^1\dots e^{r+1}},$$

$$m^{b^1\dots b^s}_{a^1\dots a^s} = \mathfrak{B}^{b^1\dots b^s}_{a^1\dots a^s},$$

$$m^{b^1\dots b^s}_{a^1\dots a^{s+r}} = -\sum_{u=2}^{r+1}\frac{u-1}{r} n^c_{c^1\dots c^u}\sum_{v=u}^{r+1} m^{e^1\dots e^{s+v-1}}_{a^1\dots a^{s+r}}\,\mathfrak{F}^{d^1\dots d^v,b^1\dots b^s}_{e^1\dots e^{s+v-1},d}\,\overset{*}{\mathfrak{K}}{}^{d,c^1\dots c^u}_{d^1\dots d^v,d},$$

$$n^a_{a^1\dots a^{r+2}} = -\mathfrak{D}^{a,b^1\dots b^{r+2}}_{(p-r-1)\,a^1\dots a^{r+2},b}\sum_{s=p-r-1}^{p}\overset{*\,bc^1\dots c^{p-r-1}}{\underset{d^1\dots d^s}{m}}\left(M^{d^1\dots d^s}_{b^1\dots b^{r+2}c^1\dots c^{p-r-1}} + M^{d^1\dots d^s}_{e^1\dots e^p} m^{e^1\dots e^p}_{b^1\dots b^{r+2}c^1\dots c^{p-r-1}}\right)+$$

$$+\mathfrak{D}^{a,b^1\dots b^{r+2}}_{(p-r-1)\,a^1\dots a^{r+2},b}\sum_{v=2}^{r+1}\frac{v-1}{r+1} n^f_{f^1\dots f^v}\sum_{u=v}^{r+2}\mathfrak{F}^{e^1\dots e^u,d^1\dots d^{p+2-u}}_{b^1\dots b^{r+2}c^1\dots c^{p-r-1},e}\times$$

$$\times\overset{*\,bc^1\dots c^{p-r-1}}{\underset{d^1\dots d^{p+2-u}}{m}}\ \overset{*\,e,f^1\dots f^v}{\underset{e^1\dots e^u,f}{\mathfrak{K}}}+$$

$$+\mathfrak{D}^{a,b^1\dots b^{r+2}}_{(p-r-1)\,a^1\dots a^{r+2},b}\overset{*\,bc^1\dots c^{p-r-1}}{\underset{d^1\dots d^p}{m}} m^{d^1\dots d^p}_{b^1\dots b^{r+2}c^1\dots c^{p-r-1}}, \qquad 1\le r\le p-2,$$

$$n^a_{a^1\dots a^{r+2}} = -\sum_{v=1}^{p} M^{b^1\dots b^v}_{a^1\dots a^{r+2}} \overset{*a}{\underset{b^1\dots b^v}{m}}, \qquad r \geq p-1.$$

Here $\mathfrak{K}^*$, m^*, and $\mathfrak{D}_{(s)}$ are the unique solutions of the nondegenerate linear systems

$$\sum_{v=s}^{t} \mathfrak{K}^{a,e^1\dots e^v}_{a^1\dots a^t,e} \overset{*e,b^1\dots b^s}{\underset{e^1\dots e^v,b}{\mathfrak{K}}} = \mathfrak{B}^{b^1\dots b^s}_{a^1\dots a^t}\mathfrak{B}^a_b,$$

$$\sum_{v=s}^{t} m^{e^1\dots e^v}_{a^1\dots a^t} \overset{*b^1\dots b^s}{\underset{e^1\dots e^v}{m}} = \mathfrak{B}^{b^1\dots b^s}_{a^1\dots a^t},$$

$$\mathfrak{F}^{c^1\dots c^s,e^1\dots e^r}_{d^1\dots d^s\,e^1\dots e^r,c}\,\mathfrak{D}^{c,b^1\dots b^s}_{(r)\,c^1\dots c^s,b} = \mathfrak{B}^{b^1\dots b^s}_{a^1\dots a^s}\mathfrak{B}^a_b.$$

4. The following lemma has been proved, which makes it possible to construct envelopes of some geometric objects by normal objects:

If the system

$$dX^{b^1\dots b^s}_{a^1\dots a^r} = X^{b^1\dots b^s}_{ca^2\dots a^r}\omega^c_{a^1} + \dots + X^{b^1\dots b^s}_{a^1\dots a^{r-1}c}\omega^c_{a^r} - X^{cb^2\dots b^s}_{a^1\dots a^r}\omega^{b^1}_c - \dots -$$

$$\dots - X^{b^1\dots b^{s-1}c}_{a^1\dots a^r}\omega^{b^s}_c + \sum_{t=2}^{w} X^{b^1\dots b^s,c^1\dots c^t}_{a^1\dots a^r,c}\ \omega^c_{c^1\dots c^t} + X^{b^1\dots b^s}_{a^1\dots a^r;c}\omega^c, \qquad \omega^c = 0,$$

in which $\omega^a_{a^1\dots a^v}$ are invariant forms of the group of analytic transformations of the parametrization, $s \neq r$, and the coefficients $X^{b^1\dots b^s,c^1\dots c^t}_{a^1\dots a^r,c}$ are known functions of the components of the normal object $\{n\}_w$, is completely integrable, then the set of functions

$$X^{b^1\dots b^s}_{a^1\dots a^r} = \frac{1}{r-s}\sum_{t=2}^{w}(t-1)n^c_{c^1\dots c^t}\sum_{v=t}^{w} X^{b^1\dots b^s,e^1\dots e^v}_{a^1\dots a^r,e}\ \overset{*e,c^1\dots c^t}{\underset{e^1\dots e^v,c}{\mathfrak{K}}}$$

is its unique solution depending only on the components of the normal objects.

5. An invariant furnishing of the surface is given in the form of the linear space $\{e_{a_1a_2}, \dots, e_{a_1\dots a_{p-1}}, M^{c_1\dots c_p}_{a_1\dots a_p}e_{c_1\dots c_p}\}$, where $e_{a_1\dots a_r} = T^\alpha_{a_1\dots a_r}E_\alpha$,

$$T^\alpha_{a_1\dots a_r} = \sum_{s=}^{r} m^{c_1\dots c_s}_{a_1\dots a_r}\Lambda^\alpha_{c_1\dots c_s}.$$

It is shown that every tensor embraced by the fundamental object of the surface is embraced by the tensors $T^\alpha_{a_1\dots a_r}$.

6. The following property (see (3)) of the tensors is indicated:

$$T^{b_1\dots b_s}_{a_1\dots a_{p+1}} \equiv \frac{1}{K} T^{\alpha}_{a_1\dots a_{p+1}} \sum_{r=s}^{p} {}^*m^{b_1\dots b_s}_{c_1\dots c_r} V^{c_1\dots c_r}_{\alpha}, \qquad s = 1, 2, \dots, p,$$

$$Q^a_{a_1\dots a_r, b} = \sum_{t=1}^{r-1} {}^*m^a_{e_1\dots e_t} m^{e_1\dots e_t}_{a_1\dots a_r; b} + \sum_{t=2}^{r+1} {}^*m^a_{e_1\dots e_{t-1} b} m^{e_1\dots e_{t-1}}_{a_1\dots c^1}, \qquad r = 2, 3, \dots, p:$$

the surface on which these tensors are equal to zero, and only such a surface, has the form

$$\vec{\Lambda} = c + \sum_{r=1}^{p} \frac{1}{r!} c_{a_1\dots a_r} s^{a_1} \cdots s^{a_r}.$$

Here $c, c_{a_1\dots a_r}$ are constant vectors, and the vectors $c_{a_1\dots a_r}$ are symmetric; the vectors $c_{a_1}, \dots, c_{a_1\dots a_{p-1}}$ with different combinations of indices are linearly independent; the rank of the collection of vectors $c_{a_1}, \dots, c_{a_1\dots a_p}$ is equal to n, while otherwise the vectors $c, c_{a_1\dots a_r}$ are arbitrary.

7. Invariants

$$\overset{\alpha}{H}_{a^1\dots a^r},$$

are constructed which possess the following property: every invariant embraced by the fundamental object of the surface is a function of them. This is done with the aid of the tensor

$$a_{ab} \equiv T^{c^1 c^2 d^1 \dots d^{p-2}}_{a e^1 e^2 d^1 \dots d^{p-2}} T^{e^1 e^2 b^1 \dots b^{p-2}}_{b c^1 c^2 b^1 \dots b^{p-2}};$$

of the tensor a^{ab}, whose components are the reduced minors of the elements of the matrix $\|a_{ab}\|$; of the tensors

$$a^a \equiv a^{ac} T^{c^1 c^2 e^1 \dots e^{p-2}}_{c b^1 b^2 e^1 \dots e^{p-2}} a^{b^1 b^2} a_{c^1 c^2}, \qquad a^b_a \equiv T^{b e b^1 \dots b^{p-2}}_{a c^1 c^2 b^1 \dots b^{p-2}} a^f a^{c^1 c^2} a_{ef};$$

of the tensors $t^a_1, t^a_2, \dots, t^a_m$, which are obtained as the result of the action of powers of the affinor a^b_a on the tensor a^a, and of the tensors

$$\Pi^{\beta}_{a_1\dots a_r} \equiv T^{\beta}_{b_1\dots b_r} t^{b^1}_{a_1} \cdots t^{b^r}_{a_r}$$

in the following way:

$$\overset{\alpha}{H}_{a^1\dots a^r} = \Pi^{\beta}_{a^1\dots a^r}\Pi^{\alpha}_{\beta}, \qquad r = 1, 2, \dots$$

Here Π^{α}_{β} are the reduced minors of the elements of the matrix of components of the tensors

$$\Pi^{\alpha}, \dots, \Pi^{\alpha}_{a_1}, \dots, \Pi^{\alpha}_{a^1\dots a^{p-1}}$$

and q tensors $\Pi^{\alpha}_{\hat{\beta}}$ from among the tensors

$$T^{\alpha}_{c^1\dots c^p} M^{c^1\dots c^p}_{b^1\dots b^p} t^{b^1}_{a_1} \cdots t^{b^p}_{a_p}.$$

(The nondegeneracy of this matrix, and also of the matrix $\|a_{ab}\|$, is verified in the case when q is either equal to m_p or is a divisor of m.)

8. The vectors

$$e_{a^1\dots a^r} \equiv \Pi^{\alpha}_{a^1\dots a^r} E_{\alpha}, \qquad r = 1, 2, \dots, p-1, \qquad e_{\hat{\alpha}} \equiv \Pi^{\beta}_{\hat{\alpha}} E_{\beta}$$

are invariant. Their totality can be used as a canonical frame of the surface.

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Note: Figure translations are in progress. See original paper for figures.