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Abstract

Full Text

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Some Questions on the Computation of Polynomials

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Mathematics

I. There exist various methods for computing the values of polynomials (both complex and real), for example Horner' s scheme*, which uses n multiplication operations and n addition operations on the value of the argument and the coefficients of the polynomial. From the point of view of speed of computation, Horner's scheme is advantageous (and so far, apparently, the most advantageous) in the case when the values of a polynomial are computed for a few values of the variable. If, however, the values of one polynomial are computed for a large number of values of the argument (for example, in computing machines one often seeks, with the required accuracy, the values

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \cdots + (-1)^{k-1}\frac{x^{2k}}{2k!},$$

then it is natural to seek methods of processing its coefficients (possibly in a complicated way, but once and for all) so as to find the values of the polynomial by means of as small a number as possible of arithmetic operations performed on the value of the variable and on the parameters—the results of the preliminary processing**.

II. The mathematical formulation of the question is as follows:

Let

$$\begin{gathered} F_n(x) = \sum_{i=0}^{n} a_i x^i; \\ p_1 = R_1' \circ R_1'', \\ p_2 = R_2' \circ R_2'', \\ \cdots\cdots\cdots \\ P_n(x) = p_k = R_k' \circ R_k'', \end{gathered} \tag{1}$$

where the sign $\circ$ denotes either $\times$ or :, or $+$ or $-$, under the following conditions:

1)

$$R_i',\ R_i'' = \begin{cases} x; \\ \alpha_1, \alpha_2, \dots, \alpha_r; \\ \text{a constant;} \\ p_\nu,\ \text{where } \nu < i; \end{cases}$$

2)

$$\alpha_j = \varphi_j(a_0, a_1, \dots, a_n) \quad (j = 1, \dots, r).$$

Then, if

$$F_n(x; a_0, \dots, a_n) \equiv P_n(x; \alpha_1, \dots, \alpha_r),$$

we shall call $P_n(x)$ a **scheme for computing the values of a polynomial with preliminary processing of the coefficients** (condition 2) defines this processing). The scheme is characterized by the number of operations $\circ$ and by the relative number of multiplicative and additive operations.

* The form of this scheme: $F_n(x) = a_0 + x(a_1 + \dots + x(a_{n-1} + xa_n)\cdots).$

** This path was first indicated by Motzkin ([1,2]).

Three questions arise:

1. Do there exist schemes of this kind with a smaller number of arithmetic operations than Horner' s scheme?
2. What is the lower bound for the number of multiplication and division operations (separately—addition and subtraction operations) in schemes of the form (1)?
3. How broad is the class of polynomials of degree n that can be computed by an "individual" scheme of type (1), i.e., by scheme (1) without a possible equality of R_i' and R_i'' to the parameters $\alpha_1, \alpha_2, \dots, \alpha_r$ in condition 1) and without condition 2), using a smaller number of multiplication and division operations (separately—addition and subtraction operations) than can be established by the answer to question 2*?

III. Question 1 is answered by the following theorem:

Theorem 1.** *For every n there exists a scheme of the form (1) using*

$$\left[\frac{n+1}{2}\right] + 1$$

multiplication operations and $n+1$ addition operations.

The form of this scheme is:

$$s_1 = x(x + \alpha_1),$$

$$s_2 = (s_1 + x + \alpha_2)(s_1 + \alpha_3) + \alpha_4,$$

$$s_3 = s_2(s_1 + \alpha_5) + \alpha_6,$$

$$\dots\dots\dots\dots\dots\dots$$

$$s_{k+1} = s_k(s_1 + \alpha_{2k+1}) + \alpha_{2k+2} \qquad (k \geq 2);$$

put (where n is the degree of the polynomial $F_n(x)$):

$$\begin{array}{ll} \text{for } n = 1 & P_1(x) = \alpha_1 x + \alpha_2, \\ \text{for } n = 2 & P_2(x) = \alpha_2 s_1 + \alpha_3, \\ \text{for } n = 3 & P_3(x) = \alpha_3 x(s_1 + \alpha_2) + \alpha_4, \\ \text{for } n = 2l \geq 4 & P_n(x) = \alpha_{n+1} s_l, \\ \text{for } n = 2l + 1 \geq 5 & P_n(x) = \alpha_{n+1} x s_l + \alpha_n; \end{array}$$

thus the scheme $P_n(x)$ is defined for all n, and the number of operations satisfies the condition of the theorem***.

The proof of Theorem 1 consists in finding such functions $\varphi_1, \varphi_2, \dots, \varphi_{n+1}$ that, for $\alpha_j = \varphi_j(a_0, \dots, a_n)$ $(j = 1, \dots, n + 1)$, we have

$$P_n(x, \alpha_1, \dots, \alpha_{n+1}) \equiv \sum_{i=0}^{n} a_{ix}^i,$$

or, in other words, in solving the system readily obtained by performing all operations in the scheme on the argument and parameters and then equating the polynomials

$$F_n(x) = \sum_{i=0}^{n} a_{ix}^i \quad \text{and} \quad P_n(x) = \sum_{i=0}^{n} \psi_i(\alpha_1, \dots, \alpha_{n+1}) x^i,$$

i.e. the system

$$a_i = \psi_i(\alpha_1, \dots, \alpha_{n+1}) \qquad (i = 0, 1, \dots, n), \tag{2}$$

where $\psi_0, \psi_1, \dots, \psi_n$ are polynomials in $\alpha_1, \dots, \alpha_{n+1}$.

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* An example of such an individual scheme for

$$F_{2^m-1}(x) = \sum_{i=0}^{2^m-1} x^i :$$

$$p_1 = x \cdot x = x^2, \quad p_2 = x^2 \cdot x^2 = x^4, \dots, \quad p_m = x^{2^{m-1}} \cdot x^{2^{m-1}} = x^{2^m},$$

$$p_{m+1} = p_m - 1, \quad p_{m+2} = x - 1, \quad P_{2^m-1}(x) = p_{m+3} = p_{m+1} : p_{m+2}.$$

** Here and below polynomials over the field of complex numbers are considered.

*** The proposed scheme is a simple generalization of an analogous scheme for $n = 6$, described by Todd (2) and belonging to Motzkin (1), coinciding with the latter for $n = 6$.

The theorem is proved by induction, first for even and then for odd n, and is a simple consequence of Lemma 1.

Lemma 1*. For every

$$f_{2n+2}(x) = \sum_{i=0}^{2n+2} a_i x_i$$

there exist $b_0, b_1, \dots, b_{2n}, \alpha_1, \alpha, \beta$ such that

$$f_{2n+2}(x) \equiv g_{2n}(x)[s_1(x) + \alpha] + \beta,$$

where

$$g_{2n}(x) = b_0 + b_1 x + \dots + b_{2n} x^{2n}; \qquad s_1(x) = x(x + \alpha_1);$$
$$a_{2n+1} = (n+1)\alpha_1 + 1; \qquad b_{2n-1} = n\alpha_1 + 1.$$

It is necessary to note that when $a_0, a_1, \dots, a_n$ are real, the system (2), generally speaking, is not solvable in real numbers.

IV. Question 2 is answered by the following theorem.

Theorem 2. For every n, every scheme $P_n(x)$ contains at least

$$\left[\frac{n+1}{2}\right]$$

multiplication and division operations and at least n addition and subtraction operations.

The theorem rests on the following simple lemma:

Lemma 2. Let R be the parameter space of the given scheme $P_n(x)$, of dimension r. Then the set A of points of this space whose coordinates $\alpha_1, \alpha_2, \ldots, \alpha_r$ satisfy, for some set $a_0, a_1, \ldots, a_n$, condition 1) of the given scheme, has dimension k, where

$$r \geq k \geq n + 1.$$

The idea of the proof of the theorem itself is that introducing more than two new parameters into the scheme with each new operation $\times$ or $:$ (or more than one parameter for operations $\pm$) is inexpedient, since the parameters introduced "above the norm" will depend on the remaining ones, and all of them together, if μ denotes the number of operations $\times$ and $:$ (respectively, σ the number of operations $\pm$), will form in parameter space a $(2\mu+1)$-dimensional (respectively, $(\sigma+1)$-dimensional) analytic surface, whence, by Lemma 2,

$$2\mu + 1 \geq n + 1, \quad \text{i.e.} \quad \mu \geq \left[\frac{n+1}{2}\right]$$

(respectively,

$$\sigma \geq n).$$

The solution of question 3 is obtained as a simple consequence of the constructions used in the proof of Theorem 2.

The final result is as follows:

In the space A of coefficients of polynomials of degree $\leq n$ there exists a finite number $r(M)$ of analytic surfaces of dimension $\leq n$, containing all polynomials with "individual schemes" in which

$$\mu < \left[\frac{n+1}{2}\right]$$

and $\sigma < M$, where $M > 0$ is arbitrary but finite (the same is true for polynomials with the number of operations $\sigma < n$ and $\mu < M$).

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REFERENCES

1. T. S. Motzkin, *Bull. Am. Math. Soc.*, **61**, No. 1, 163 (1955).

2. J. Todd, *Comm. Pure and Appl. Math.*, **8**, No. 1 (1955); Russian transl.: J. Todd, *Mathematical Education*, issue 1 (1957).

* First formulated in this form by V. Ya. Pan.

Note: Figure translations are in progress. See original paper for figures.