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Abstract

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AN EXACT INTEGRAL OF THE FIELD EQUATION FOR WIND WAVES IN THE OCEAN AND ITS PHYSICAL SIGNIFICANCE

The approximate integration of the field equation for wind waves in the ocean ([1,2]) and of an analogous equation derived for shallow seas (([3]) and ([2]), p. 103) made it possible for us to trace the distribution of wave heights over the distance from the windward shore to the leeward shore and over the course of various intervals from the beginning of the wind' s action. Even the approximate solutions agreed satisfactorily with the results of direct measurements of waves under natural conditions. The basic differential equations were written by us in dimensionless form, with subsequent determination of the scales from the prescribed particular conditions ([4]). As applied to waves in the ocean, it was established that the dimensionless wave height η is related to the dimensionless time τ and the dimensionless distance ξ from the windward shore (or the windward boundary of a storm area in the ocean) by the equation

$$\partial\eta/\partial\tau = 1 - \eta - \eta^{1/2}\partial\eta/\partial\xi. \quad (1)$$

Recent investigations of equations of this type (the so-called quasi-linear equations), carried out by A. N. Tikhonov and A. A. Samarskii ([5]), make it possible to approach equation (1) in a new way and to show that its exact integral is fully determined by those exact solutions which we previously found as applied to two simple conditions: a) to the distribution of the heights of established waves at various distances ξ from the windward shore; b) to the increase in wave height η over the duration of wind action τ at a very large distance from the windward shore (theoretically infinite). On the basis of the well-known relation between logarithms and inverse hyperbolic functions, for condition "a" one may write, following ([1,2]),

$$\xi = 2\,\mathrm{Ar\,th}\,\eta^{1/2} - 2\eta^{1/2}. \quad (2)$$

For condition "b," the work ([1,2]) gives

$$\eta = 1 - e^{-\tau}. \quad (3)$$

Fig. 1

Figure 1: Fig. 1

The mathematical investigations ([5]) show that, in a certain plane in the rectangular coordinate system τ, ξ, there must exist a discontinuity line $\xi(\tau)$: on one side of it condition "a" is satisfied, owing to the fact that here $\partial\eta/\partial\tau = 0$; on the other side condition "b" comes into force, since here $\partial\eta/\partial\xi = 0$. According to the method ([5]), one seeks the differential equation of the discontinuity line by passing from $\Delta\xi/\Delta\tau$ to the limit $d\xi/d\tau$. Let us note here that the last term on the right-hand side of (1) may be represented as $\frac{\partial}{\partial\xi}\left[{}^2/{}_3\eta^{3/2}\right]$. Denote the function η near the discontinuity line at one end of a very small segment $\Delta\xi$ by $\eta(\tau)$, and at the other end of the segment by $\eta(\xi)$. Then, by the method ([5]), in the limit one obtains

$$\frac{d\xi}{d\tau} = \frac{2}{3}\,\frac{\eta(\tau)^{3/2} - \eta(\xi)^{3/2}}{\eta(\tau) - \eta(\xi)}. \tag{4}$$

On the basis of natural physical considerations, let us write that the discontinuity begins its motion over the surface of the ocean from the windward shore itself (or from the windward boundary of the storm). Then it will be necessary to assume that at $\tau = 0$, $\xi = 0$. In addition, in formula (4), for infinitely small τ and ξ, the numerator is an infinitesimal of higher order in comparison with the denominator. Hence, $d\xi/d\tau = 0$ at $\tau = 0$.

Dividing the time τ into small intervals equal to 0.1, we integrated (4) by the approximate numerical Euler-Cauchy method[6]. It turned out that, on approaching the line found from the left and from the right, only $\partial\eta/\partial\tau$ and $\partial\eta/\partial\xi$ undergo a discontinuity, while η changes continuously. This creates a second—more accurate—method of construction, which is based on the physical meaning of the phenomena:

1. At the initial stage of the generation of wind waves, even very near the windward shore, equality cannot yet be said to hold between the fluxes of energy brought into and carried out of some unit vertical prism in the water mass[1,2]; therefore, practically throughout the entire extent of the sea, the waves may develop in accordance with equation (3): the expenditure of energy per unit sea surface is caused only by internal turbulent friction in the water.

Fig. 1

2. Developing with time according to the law (3), the waves at some distance ξ_k from the windward shore will have time to grow only to that height η_k which corresponds to the steady wave motion at the given distance ξ_k.

This will occur after a time τ_k, which corresponds to the ordinate ξ_k on the sought discontinuity line $\xi(\tau)$, drawn in the coordinate system τ, ξ.

3. After this, over the extent ξ_k from the windward shore there will exist a steady wave motion, and the extent ξ_k itself will increase continuously with time, since farther and farther from the windward shore there will begin to occur a kind of "saturation."

4. As a result, the discontinuity line $\xi(\tau)$ will describe the law of the gradual advance of a peculiar "front" from the windward shore into the open sea. Behind such a "front" there will be a steady wave motion characterized by equation (2). Ahead of the "front," the wave height will grow according to the law described by equation (3).

5. At points lying on the "front" itself, the wave height must be determined unambiguously both from (2) and from (3), and this is accomplished when ξ in (2) and τ in (3) serve as the coordinates of one and the same point on the auxiliary diagram—on the discontinuity line $\xi(\tau)$.

It is precisely on this basis that the line $\xi(\tau)$ in Fig. 1 was constructed. Specifying a definite value of τ, we found from (3) the corresponding values of η. Substituting the obtained value of η into (2), we determined the distance ξ to the windward shore at which η corresponds to the steady wave motion. In other words, in Fig. 1 the selected values of the time τ, during which the "front" moved into the open sea, are plotted along the abscissa axis, while along the ordinate axis are plotted the distances ξ over which the "front" managed to move away from the windward shore during the time τ. The curve $\xi(\tau)$, constructed in this way, coincides with that found by numerical integration of (4).

Having the auxiliary curve of Fig. 1, let us construct, in dimetric projection, the surface $\eta(\xi, \tau)$, which will give a geometric representation of the exact integral of equation (1): in Fig. 2, where integral values of the corresponding quantities are placed along the ξ and τ axes, and the intervals are divided into segments of 0.2; the same segments are marked by points on the η axis. As can be seen from Fig. 2, this is a very interesting surface, decomposing into two cylindrical surfaces that intersect along the line of double curvature OK. If we make a vertical straight line (parallel to the η axis) slide along the line OK, we obtain a third cylindrical surface, intersecting the coordinate plane $\xi\tau$ along the curve OK'. It is easy to see that OK' is the auxiliary curve $\xi(\tau)$, transferred here from Fig. 1 (but with a different orientation of the axes ···)

τ, ξ. Thus, the surface $\eta(\xi, \tau)$ found in Fig. 2, with a break along the line OK, gives a complete representation of the physical meaning of the dimensionless wave heights η: to the left of the edge OK lies the region of established waves, and to the right of OK the region of waves increasing with time. At all distances from the windward shore ξ, the increase of η follows one and the same law (3), but it ends the sooner the smaller ξ is; with further increase of τ, the height η remains unchanged, as is shown by the lines parallel—

Fig. 2

Figure 2: Fig. 2

Fig. 2

to the τ-axis in Fig. 2. The law of distribution of the heights of established waves (2) is just as universal: all vertical planes in Fig. 2 parallel to the ξ-axis give identical curves when intersected with the surface $\eta(\xi, \tau)$, but for different values of the duration of the wind action τ, the segments of these curves from the τ-axis to the edge OK are not the same: the longer the wind has acted, the farther from the shore the region of established wave motion has spread; beyond its limits, to the right of the edge OK, all the heights of the growing waves at that moment are identical, as is shown in Fig. 2 by straight lines parallel to the ξ-axis. It is easy to verify that at all points of the surface $\eta(\xi, \tau)$ found in Fig. 2 the differential equation of the wave field (1) is satisfied.

Up to this point we have here investigated the behavior of the dimensionless function η under changes of the two dimensionless arguments ξ, τ. On the basis of our work [4], a transition is made from these quantities (which are very convenient for investigation) to the quantities of greatest practical importance. Namely, η is the ratio of the wave height at a given moment at a given point of the ocean to the greatest height that can arise at a given wind speed V at an extremely large distance from the windward shore and after a very long duration of action of this wind (theoretically infinite).

The relation between the true distance x from the windward shore and the dimensionless quantity ξ is determined by the scale relation (5) from work [4]. The relation between the true duration of the wind action t and the dimensionless quantity τ is given in the scale relation (9) in the same work [4]. It is also shown there how the second approximation is found in calculating the wave height at various points of the ocean at various times. We shall now use relations (5) and (9) from [4] for another purpose: we shall elucidate fully the physical meaning

motion of our "front" from the shore into the open ocean. To this end, let us first of all return to Fig. 1, where the curve $\xi(\tau)$ shows how, as the waves develop, the influence of the windward shore gradually propagates farther and farther or, in short, how the "coastal effect" begins to make itself felt farther and farther away.

The speed of advance of the "front" was already specified by formula (4), which served for the approximate integration. For reliability we also determined $d\xi/d\tau$ from the curve $\xi(\tau)$ plotted in Fig. 1. As a result, the curve $d\xi/d\tau$ was obtained, also plotted in Fig. 1 and showing that already after the lapse of the time $\tau = 1.2$ the value $d\xi/d\tau$ reaches values exceeding 0.9, and then slowly tends to the limiting value $(d\xi/d\tau)_\infty = 1$.

On the other hand, on the basis of (5) and (9) from (4) one can, from the known value $d\xi/d\tau$, determine the practically important quantity dx/dt—the true speed of advance of the "front." This speed at first increases rapidly, and then slowly. We shall determine it only as applied to the final stage of wave development, at which the scale ratios x/ξ and t/τ change very slowly. In the numerator and in the denominator of the expression obtained on the basis of (5) and (9) from (4), there will be common factors that have entered there from the indicated relations. After canceling these common factors we obtain

$$\frac{dx}{dt} = 0.895\,\frac{2\pi}{9}\,\frac{d\xi}{d\tau}\,f_\infty V. \tag{5}$$

Here f_∞, according to (4), denotes the quotient obtained by dividing the phase speed of the largest waves possible at wind speed V by the wind speed V itself. Hence $f_\infty V = c$.

Substituting into (5) the limiting value found, $d\xi/d\tau = 1.0$, we finally have:

$$dx/dt = 0.625c. \tag{6}$$

It is probably not necessary to attach special significance to the value 0.625 in (6): there are grounds for believing that it differs from 1/2 only because in all the original equations of the energy balance of the waves, for the solvability of the problem in quadratures, we substituted the classical expression for wave energy. In reality, however, on the basis of the investigations of A. I. Nekrasov[7] and our work[8], it must be assumed that at the final stage of wave development their energy exceeds by 25% the value obtained from the classical formula. On the other hand, the flux of energy carried by the waves is equal to the product of the wave energy and one half of their phase speed. It is precisely the variability of this flux with distance from the windward shore that gives rise to the last term on the right-hand side of (1). Keeping the former value of the energy flux, we reduce by 25% the factor multiplying c in (6). Then it turns out that the speed of advance of the "front" is equal to the group velocity of the waves. It is precisely with the group velocity of the waves that the "coastal effect" advances here into the ocean.

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Note: Figure translations are in progress. See original paper for figures.