THE EMBEDDING PROBLEM FOR SPLIT EXTENSIONS

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Abstract

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THE EMBEDDING PROBLEM FOR SPLIT EXTENSIONS

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The embedding problem assumes as given a normal extension k/Ω with Galois group F, a group G, and an epimorphism $\varphi : G \to F$. It is required to find conditions under which there exists a normal extension K/Ω with Galois group G such that $K \supset k$ and the epimorphism φ coincides with the natural homomorphism of the Galois group of the field onto the Galois group of the subfield.

The group G is called a **split extension of its image** F **under the homomorphism** φ if it contains a subgroup which is mapped isomorphically onto F under φ. There may be several such subgroups. In what follows we shall consider one of them to be chosen and shall denote it by F. If the kernel of φ is N, then $G = F \cdot N$. We shall say that G is a **split extension of the group** F **with kernel** N.

The purpose of the present note is to announce the following result:

Theorem 1. *The embedding problem is solvable for any field of algebraic numbers k, if G is a split extension with nilpotent kernel.*

This theorem contains, as special cases, a number of results obtained earlier in the embedding problem and in the problem of constructing fields with a prescribed Galois group.

If $F = 1$, then Theorem 1 proves the existence of an extension with arbitrary nilpotent Galois group N, which had been proved, when the order of N is odd, by Scholz ([1]) and Reichardt ([2]), and for an arbitrary group N by the author ([3]). In the case where the kernel N is abelian, Theorem 1 was proved by Scholz ([4]) and, by a more direct method, by Delone and Faddeev ([5]). In the cases where the group N is a p-group of class $\leq p$ or the orders of the groups G and N are relatively prime, Theorem 1 was proved by the author ([6]). Since every solvable group $\mathfrak{G}$ is a quotient group of a group G obtained by a chain of split extensions with nilpotent kernels ([7]), Theorem 1 implies the existence of a field of algebraic numbers with an arbitrary solvable Galois group. This fact was earlier proved by the author ([8]) on the basis of the solution of a certain more artificial embedding problem.

The proof of Theorem 1 is based on considerations close to those used in the author' s papers ([3,6,8]). The main difference is that the notion of a Scholz field used in those papers is now replaced by the notion of a relatively Scholz field. It is obvious that one may restrict oneself to the case where N is a group of order l^α, where l is a prime number. Let us first consider the case where G is also an l-group. The subfield of the field K belonging to the subgroup F will be denoted by L.

We shall call the field K relatively Scholz (relative to k) if the following conditions are satisfied for it:

1. From each prime divisor of the discriminant of the field k/Ω there splits off in L a prime factor of the first order in the first degree.
2. The prime divisors of the discriminant K/k decompose in K into prime divisors of degree 1, and in k/Ω are not critical and have order 1.
3. The absolute norms of the prime divisors of the discriminant K/k satisfy the conditions

$$\mathfrak{N}(\mathfrak{p}) \equiv 1 \pmod{l^h},$$

where h is sufficiently large.

4. The prime divisors l split completely in K/k.
5. The real infinitely remote divisors of k remain real in K.

The connection of this notion with the embedding problem is based on the following theorem. Let Z be a normal divisor of order l in N. Denote by $\overline{N}$, $\overline{G}$, and $\overline{K}$ the groups N/Z, G/Z, and the field having Galois group $\overline{G}$. The homomorphism $\varphi : G \to \overline{G}$ maps F isomorphically. Identifying φF with F, we may write that $\overline{G} = \overline{N} \cdot F$. The subfield of $\overline{K}$ belonging to $\overline{N}$ will be denoted by k.

Theorem 2. *The embedding problem determined by the field $\overline{K}$ and the homomorphism $\varphi : \overline{G} \to G$ is solvable if the field $\overline{K}$ is relatively-Scholz (relative to k).*

Let now the order of G be arbitrary. Denote by H the Sylow l-subgroup of G. Let A be the minimal Abelian normal divisor of G lying in N. Put $G/A = \overline{G}$, $H/A = \overline{H}$, $N/A = \overline{N}$, and denote by φ the homomorphism of G onto $\overline{G}$. The field with Galois group $\overline{G}$ over the field $\varkappa$ will be denoted by $\overline{K}/\varkappa$, and its subfields belonging to $\overline{H}$ and $\overline{N}$ by Ω and k. Applying the reduction theorem of Faddeev ([9]) and Kochendörffer ([10]), one can obtain from Theorem 2 the following result.

Theorem 3. *The embedding problem determined by the field $\overline{K}/\varkappa$ and the homomorphism $\varphi : G \to \overline{G}$ is solvable if the field $\overline{K}/\Omega$ is relatively-Scholz (relative to k).*

The field K which is a solution of the embedding problem formulated in Theorem 3 will not, in general, be relatively-Scholz. Conditions for it to be possible to choose it to be relatively-Scholz can be found analogously to the way this was done in the works ([3,6,8]). They consist in the equality to one of certain invariants $\chi_i(\overline{K})$ of the field $\overline{K}$, whose values are roots of l-th degree of 1.

Suppose that the group N has d generators $s_1, \ldots, s_d$, and that the group F has order m. Consider the reduced free group of class c, $\mathfrak{M}_d^c$, with md generators $\sigma_{f,i}$, $i = 1, \ldots, d$, $f \in F$, whose factor group is N. Defining the permutation f with $\sigma_{f,i}$ by the rules

$$f_1^{-1} \sigma_{f,i} f_1 = \sigma_{ff_1,i},$$

we obtain the group $F \cdot \mathfrak{M}_d^c$, which we shall denote by $\mathfrak{G}_d^c$. The mapping

$$f \to f, \qquad \sigma_{f,i} \to s_i^f$$

defines a homomorphism of $\mathfrak{G}_d^c$ onto G, mapping F isomorphically onto F. It follows from this that the solvability of the embedding problem determined by the field k and the homomorphism $\mathfrak{G}_d^c \to F$ implies the solvability of the original embedding problem.

Suppose that for every d the existence has been proved of a field $K_d^{c-1} \supset k$ having Galois group $\mathfrak{G}_d^{c-1}$. For the possibility of embedding this field into the field K_d^c, it is necessary that the invariants $\chi^i(K_d^{c-1})$, mentioned above, be equal to 1. Applying the apparatus developed in the works ([3,6,8]), one can obtain the following assertion.

Theorem 4. For every natural number δ there exists a $d(\delta)$ such that in any field $\overline{K_\delta^{c-1}}$ containing k and relatively Scholzian, there exists a subfield $\overline{K_\delta^{c-1}}$, also containing k, whose invariants $\chi_i(\overline{K_\delta^{c-1}})$ are equal to 1.

Such a field $\overline{K_\delta^{c-1}}$ can therefore be embedded in a relatively Scholzian (over k) field K_δ^c. The successive application of this process gives the proof of Theorem 1.

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CITED LITERATURE

[1] A. Scholz, Math. Zs., **42**, 161 (1936).
[2] H. Reichardt, J. Reine u. Angew. Math., **177**, 1 (1937).
[3] I. R. Shafarevich, Izv. AN SSSR, ser. matem., **18**, 216 (1954).
[4] A. Scholz, Math. Zs., **30**, 332 (1929).
[5] B. N. Delaunay and D. K. Faddeev, Matem. sborn., **15** (57), 243 (1943).
[6] I. R. Shafarevich, Izv. AN SSSR, ser. matem., **18**, 389 (1954).

[7] O. Ore, Duke Math. J., **5**, 431 (1939).
[8] I. R. Shafarevich, Izv. AN SSSR, ser. matem., **18**, 525 (1954).
[9] D. K. Faddeev, DAN, **92**, 703 (1953).
[10] R. Kochendörffer, Math. Nachr., **10**, 75 (1953).

Note: Figure translations are in progress. See original paper for figures.