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B. A. TRUBNIKOV

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## Abstract

## Full Text

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B. A. TRUBNIKOV

# RADIATION OF PLASMA IN A MAGNETIC FIELD

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1. In this paper we consider plasma radiation caused by the rotation of electrons in a uniform magnetic field. The radiation spectrum of a fast electron moving in a magnetic field consists, as is well known ([1,2]), of separate lines —harmonics. Bearing in mind applications to the radio astronomy of the Sun and the Galaxy, we shall restrict ourselves to consideration of high overtones of the spectrum; moreover, we shall assume the real parts of the refractive indices of the medium to be equal to unity, which is obviously valid for sufficiently high frequencies. The distribution of electrons over momenta is assumed isotropic:

$$dN = f(|\mathbf{p}|)\, dp = N(E)\, dE\, \frac{dO_p}{4\pi}. \qquad (1)$$

Having determined, from the known general formula, the vector potential $\mathbf{A}_\omega$ of the field produced by an electron, for the case of motion along a helix we find the distribution of the radiation intensity ($\vec{\beta} \equiv \mathbf{v}/c$):

$$\frac{dJ(\theta)}{d\omega\, d\Omega} = \frac{e^2\omega^2}{2\pi c} \sum_1^\infty \left[ \left( \frac{\cos\theta - \beta_\parallel}{\sin\theta} \right)^2 J_n^2 + \beta_\perp^2 J_n'^{\,2} \right] \delta\left[ n\omega_0 - \omega(1 - \beta_\parallel \cos\theta) \right]. \qquad (2)$$

Here

$$J_n = J_n\left( \frac{\omega}{\omega_0} \beta_\perp \sin\theta \right),$$

$J_n'$ are Bessel functions and their derivatives;

$$\omega_0 = \left| \frac{eH_0}{mc} \right| \sqrt{1 - \beta^2}$$

is the frequency of rotation of the electron along the spiral; $\theta$ is the angle between the direction of radiation and the magnetic field $\mathbf{H}_0$. For simplicity, in what follows we restrict ourselves to the case $\theta = \pi/2$.

Then it is not difficult to establish that the term with $J_n'^2$ in formula (2) corresponds to radiation of linearly polarized (only for $\theta = \pi/2$!) waves with $\mathbf{E}_\sim \parallel \mathbf{H}_0$ ( "ordinary" wave)

$$\eta_1^{(\parallel)} \equiv \frac{dJ^{(\parallel)}(\pi/2)}{d\omega\, d\Omega} = \frac{e^2\omega^2}{2\pi c} \sum_1^\infty \beta_\perp^2 J_n'^2(n\beta_\perp)\, \delta(n\omega_0 - \omega). \tag{3}$$

The second term gives radiation of waves for which $\mathbf{E}_\sim \perp \mathbf{H}_0$ ( "extraordinary" wave):

$$\eta_1^{(\perp)} \equiv \frac{dJ^{(\perp)}(\pi/2)}{d\omega\, d\Omega} = \frac{e^2\omega^2}{2\pi c} \sum_1^\infty \beta_\perp'^2 J_n^2(n\beta_\perp)\, \delta(n\omega_0 - \omega). \tag{4}$$

After averaging these formulas over the directions of the electron velocities, we obtain

$$\bar\eta_1^{(\parallel,\perp)} = \frac{e^2\omega^2}{2\pi c} \sum_1^\infty S_n^{(\parallel,\perp)}(\gamma)\, \delta\left(\frac{n\Omega}{\gamma} - \omega\right), \tag{5}$$

where

$$\Omega = \left|\frac{eH_0}{mc}\right|, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{mc^2},$$

$$S_n^{(\parallel)}(\gamma) = \frac{1}{4\pi} \oint \beta_\parallel^2 J_n^2(n\beta_\perp)\, dO_p, \qquad S_n^{(\perp)}(\gamma) = \frac{1}{4\pi} \oint \beta_\perp^2 J_n'^2(n\beta_\perp)\, dO_p. \tag{6}$$

For large $n \gg 1$, approximate expressions of two types can be obtained for these integrals:

for $\dfrac{n}{\gamma^3} \gg 1$

$$S_n^{(\parallel,\perp)}(\gamma) = \frac{e^{2n/\gamma}}{4n\sqrt{\pi n\gamma}} \left(\frac{\gamma-1}{\gamma+1}\right)^n \left\{\frac{\gamma^2-1}{2n/\gamma}, 1\right\}; \tag{7}$$

for $n \gg 1,\ \gamma \gg 1$

$$S_n^{(\parallel,\perp)}(\gamma) = \frac{1}{4\pi\gamma^2\sqrt{3}} \left[\int_{(2n/3\gamma^3)}^\infty K_{5/3}(t)\, dt \mp K_{2/3}\left(\frac{2n}{3\gamma^3}\right)\right]. \tag{8}$$

Let us consider applications of the formulas obtained.

2. For nonrelativistic and weakly relativistic plasma temperatures, radiation of high overtones is of interest only for large system sizes, when the initial harmonics of the spectrum are found to be trapped. Such conditions are realized, for example, in the solar corona. Let us find the radiation of a plasma layer of thickness $a$; for this purpose we use the usual formula:

$$I_\omega^{(i)} = \frac{\eta_\omega^{(i)}}{\alpha_\omega^{(i)}} \left(1 - \exp\left[-\int_0^a \alpha_\omega^{(i)}\, ds\right]\right) \tag{9}$$

($i$–polarization). Here $\eta_\omega^{(i)}$ is the emissivity per unit volume of plasma; $\alpha_\omega^{(i)}$ is the absorption coefficient. It can be calculated in the following way. We shall take the electron distribution function to be Maxwellian, $f = c\exp(-E/T)$; then, using Kirchhoff' s law, one can find

$$\alpha_\omega^{(i)} = \frac{\eta_\omega^{(i)}}{I_0} = \frac{4\pi}{I_0}\int_0^\infty f\, \overline{\eta}_1^{(i)} p^2\, dp, \tag{10}$$

where $I_0 = \omega^2 T/8\pi^3 c^2$ is the equilibrium Rayleigh–Jeans flux. Hence, using formulas (5) and (7) for $\overline{\eta}_1^{(i)}$, we obtain (for $\mu \equiv mc^2/T \gg 1$)

$$\alpha_\omega^{(\parallel,\perp)} = \left(\frac{\omega_\lambda^2}{c\Omega}\right) \frac{\mu^{5/2} e^{2x}}{2^{3/2} x^{5/2}} \sum_{n>x} e^{-\mu(\frac{n}{x}-1)} \sqrt{\left(\frac{n}{x}\right)^2 - 1}\left(\frac{n/x - 1}{n/x + 1}\right)^n \left\{\frac{(n/x)^2 - 1}{2x}, 1\right\}, \tag{11}$$

where $\omega_\lambda^2 = 4\pi N e^2/m,\ x = \omega/\Omega,\ N$ is the electron density.

In computing the "optical thickness" $\int \alpha\, ds$, we shall have to integrate $\alpha$ over different regions of the plasma, in which the magnetic field may have different magnitude (we regard the direction of $\mathbf{H}$ as constant). This is equivalent to averaging over the values $|H|$, which, together with Doppler broadening, will lead to a smearing of the individual absorption lines. Therefore the spectrum may be regarded as continuous, with $H_0$ understood as the mean field. Then the sum in formula (11) may be replaced by an integral, which for $\mu \gg 1,\ x \gg 1$ we evaluate by the saddle-point method. As a result we obtain the formula

$$\alpha_\omega^{(\parallel,\perp)} \simeq \left(\frac{\omega_\lambda^2}{c\Omega}\right) \frac{\sqrt{\pi\mu}}{4} (\varepsilon^2 - 1)^{3/2} \frac{\mu^2}{x^2} e^{\mu - \frac{2x}{\varepsilon^2 - 1}} \left\{\frac{\varepsilon^2 - 1}{2x}, 1\right\}, \tag{12}$$

where $\varepsilon$ is to be determined from the equation

$$\frac{2\varepsilon}{\varepsilon^2 - 1} - \ln\frac{\varepsilon + 1}{\varepsilon - 1} = \frac{\mu}{x}. \tag{13}$$

In limiting cases the following approximations hold:

for $\frac{x}{\mu} \ll 1$

$$\alpha_\omega^{(\|,\perp)} \simeq \left(\frac{\omega_\lambda^2}{c\Omega}\right) \sqrt{\frac{\pi}{4}\,\mu e \left(\frac{ex}{2\mu}\right)^{x-1/2}} \left\{\frac{1}{\mu},\,1\right\}; \tag{14}$$

for $x = \frac{9x}{2\mu} \gg 1$

$$\alpha_\omega^{(\|,\perp)} \simeq \left(\frac{\omega_\lambda^2}{c\Omega}\right) \frac{3}{2}\sqrt{\pi\mu}\, \frac{\exp\left\{-\mu\left[x^{1/3} - 1 + \frac{9}{20x^{1/3}}\right]\right\}}{x} \left\{\frac{1}{\mu x^{1/3}},\,1\right\}. \tag{15}$$

In both cases the "extraordinary" wave ($\perp$) is absorbed considerably more strongly than the "ordinary" wave ($\|$), which should lead to polarization of the radiation.

The total flux emerging from the plasma can be estimated from the formula:

$$W^{(i)} = \int_0^\infty I_0\left(1 - e^{-\alpha a}\right) d\omega \simeq \int_0^{\omega_{\max}} I_0 d\omega = \frac{\omega_{\max}^3 T}{8\pi^3 c^2 \cdot 3}, \tag{16}$$

where $\omega_{\max}$ is determined by the relation $\alpha a \simeq 1$.

Under the conditions of the solar corona one may take, for an estimate, $H \sim 10$ gauss, $N \sim 10^7$ cm$^{-3}$, $a \sim r_\odot \sim 7 \cdot 10^{10}$ cm, $T \sim 100$ eV. Using formula (14), we find $x_{\max} = \omega_{\max}/\Omega \simeq 4$, i.e., approximately the first four harmonics are trapped.

3. Let us consider the case of ultrarelativistic electron energies, when formula (8) is valid for the integrals $S_n^{(i)}(\gamma)$. In this case the radiation spectrum may be regarded as continuous, and from formulas (5) and (8) we obtain:

$$\overline{\eta}_1^{(\|,\perp)} \simeq \frac{e^2\omega^2}{2\pi c}\frac{\gamma}{\Omega}\left(S_n^{(\|,\perp)}(\gamma)\right)_{n=\frac{\omega}{\Omega}\gamma} = A\left(\frac{\omega}{\omega_c}\right)\left[\int_{(\omega/\omega_c)}^\infty K_{5/3}(t)\,dt \mp K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right], \tag{17}$$

where the frequency $\omega_c = \frac{3}{2}\Omega\left(\frac{E}{mc^2}\right)^2$, and the constant $A = \frac{e^2\Omega\sqrt{3}}{16\pi^2 c}$.

Adding the radiation intensities of both polarizations, we find the total intensity

$$\overline{\eta}_1\left(\frac{\pi}{2}\right) = \overline{\eta}_1^{(\perp)} + \overline{\eta}_1^{(\|)} = A \cdot 2\left(\frac{\omega}{\omega_c}\right)\int_{(\omega/\omega_c)}^\infty K_{5/3}(t)\,dt. \tag{18}$$

Function (18) was considered earlier and was tabulated in ([3]); it has a maximum at $\omega/\omega_c = 0.3$. The degree of polarization of the radiation produced by electrons of one energy $E$ is found from (17):

$$\rho_e\left(\frac{\pi}{2}\right) = \frac{\overline{\eta}_1^{(\perp)} - \overline{\eta}_1^{(\|)}}{\overline{\eta}_1^{(\perp)} + \overline{\eta}_1^{(\|)}} = \frac{K_{2/3}(\omega/\omega_c)}{\displaystyle\int_{(\omega/\omega_c)}^{\infty} K_{5/3}(t)\,dt} = \begin{cases} \dfrac{1}{2}, & \text{for } \omega \ll \omega_c, \\ 1 - \dfrac{2\omega_c}{3\omega}, & \text{for } \omega \gg \omega_c. \end{cases} \tag{19}$$

If one assumes that the energy spectrum of the electrons has a power-law form $N(E) = K/E^{\nu}$, then the emissivity of a unit volume of plasma can be found:

$$\eta_\omega^{(i)} = \int_0^\infty N(E)\,\overline{\eta}_1^{(i)}\,dE = \text{const}\cdot\omega^{-(\nu-1)/2}. \tag{20}$$

For the degree of polarization we then obtain:

$$\rho_\nu\left(\frac{\pi}{2}\right) = \frac{\eta^{(\perp)} - \eta^{(\|)}}{\eta^{(\perp)} + \eta^{(\|)}} = \frac{\displaystyle\int_0^\infty (dE)\,E^{-\nu}\left(\frac{\omega}{\omega_c}\right)K_{2/3}\left(\frac{\omega}{\omega_c}\right)}{\displaystyle\int_0^\infty (dE)\,E^{-\nu}\left(\frac{\omega}{\omega_c}\right)\int_{(\omega/\omega_c)}^{\infty} K_{5/3}(t)\,dt} = \frac{\nu+1}{\nu+7/3}. \tag{21}$$

Thus, the degree of polarization turns out to be connected with the form of the spectrum.

As is known, an almost 100% polarization of the optical radiation of the Crab Nebula has recently been discovered, which confirms I. S. Shklovsky' s supposition ([4]) concerning the magneto-bremsstrahlung mechanism of its luminosity, due to relativistic electrons moving in turbulent magnetic fields. For the spectrum averaged over the whole nebula one usually assumes $\eta \sim \omega^{-1}$, which corresponds to $\nu = 3$. Then from (21) we find the maximum possible degree of polarization $\rho = 75\%$. Since higher degrees of polarization are observed in individual regions of the nebula, the spectrum emitted by these regions must fall more steeply. On the other hand, the discovery of a degree of polarization very close to 100% (which is possible by formula (19) only for $\omega_c \ll \omega$) forces one somewhat to reduce the electron energies needed to explain the observed frequencies, in comparison with the energies that are usually determined from the relation $\omega_c \sim \omega$ (by two orders of magnitude at 99%).

Let us note that these considerations are valid only under the condition that the distribution of the electrons in momenta is isotropic.

We shall now take absorption into account. Using the principle of detailed balance, we find for the absorption coefficient the formula

$$\alpha_\omega^{(i)} = \frac{8\pi^3 c^2}{\omega^2} \int \frac{f(E) - f(E + \hbar\omega)}{\hbar\omega} \eta_1^{(i)} \, d\mathbf{p}. \tag{22}$$

In the case of small quantum energies,

$$\lim_{\hbar\to 0} \frac{f(E) - f(E + \hbar\omega)}{\hbar\omega} = -\frac{\partial f}{\partial E}, \qquad \alpha_\omega^{(i)} = \frac{8\pi^3 c^2}{\omega^2} \int \left(-\frac{\partial f}{\partial E}\right) \eta_1^{(i)} \, d\mathbf{p}. \tag{23}$$

For a power-law spectrum $N(E) = K/E^\nu$ of ultrarelativistic electrons, we obtain

$$\alpha_\omega^{(i)} = \frac{8\pi^3 c^2}{\omega^2} \int_0^\infty E^1 \left[-\frac{\partial}{\partial E}\left(\frac{N(E)}{E^2}\right)\right] \overline{\eta_1^{(i)}} \, dE = \text{const} \cdot \omega^{-(\nu/2+2)}. \tag{24}$$

Using formula (20) for $\eta_\omega^{(i)}$, we find $\eta/\alpha = C\omega^{5/2}$. Thus, for the radiation of a layer there must be the spectral distribution

$$I_\omega = \frac{\eta}{\alpha}\left(1 - e^{-\alpha a}\right) = C\omega^{5/2}\left[1 - \exp\left(-\frac{\text{const}}{\omega^{2+\nu/2}}\right)\right]. \tag{25}$$

As the frequency is decreased, when the "optical thickness" of the layer $\alpha a$ becomes of the order of unity or larger, this spectrum begins to fall steeply. This, apparently, can explain the behavior of the spectrum of the powerful source of cosmic radio emission Cassiopeia A in the frequency region of 20 Mc/s.

In conclusion we note that a kinetic treatment with the introduction of the electric permittivity $\varepsilon$ leads to formulas analogous to those obtained in the present work, if one assumes that the polarizability of the medium is small, i.e. $|\varepsilon - 1| \ll 1$. The latter condition turns out to be satisfied for the high harmonics considered.

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*Note: Figure translations are in progress. See original paper for figures.*