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Abstract

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ON CERTAIN TYPES OF HOMOGENEOUS RIEMANNIAN SPACES

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1. Let V_n be a homogeneous Riemannian space with positive-definite metric, G its group of motions, and H the stationary group of a given point P_0. If E is the tangent space at this point, then the group H may be regarded as a group of rotations (the isotropy group) in E.

In the work of G. I. Kruchkovich and the author [1] the following theorem was obtained:
If the isotropy group H decomposes into the direct product of two subgroups $H = H_0 \times H_1$, where H_1 is irreducible and admits no rotations permutable with it on the plane E_1 on which it acts in E, while H_0 acts on the orthogonal complement to E_1, then the space V_n is semi-reducible, i.e. its metric can be reduced to the form

$$ds^2 = g_{i_0 j_0}(x^{k_0})\, dx^{i_0} dx^{j_0} + \sigma(x^{k_0}) g_{i_1 j_1}(x^{k_1})\, dx^{i_1} dx^{j_1} \tag{1}$$

$$(i_0, j_0 = 1, \dots, q, \qquad i_1, j_1 = q+1, \dots, n)$$

and the group G is a non-mixing group of this space [2].

The purpose of the present note is to study such homogeneous Riemannian spaces V_n for which the isotropy group H acts irreducibly on the plane E_1 and admits there rotations permutable with it. It is assumed here that the orthogonal complement E_0 to E_1 in the tangent space E consists entirely of fixed directions of the group H.

2. Let $\{M, e_1, \dots, e_n\}$ be a system of orthonormal frames in V_n, admissible relative to the group of motions G, and

$$dM = \omega^i e_i, \qquad de_i = \omega_i^j e_j \quad (i, j = 1, \dots, n) \tag{2}$$

the equations of infinitesimal displacements of these frames; ω^i, ω_i^j are invariant forms of the group G. The Maurer-Cartan equations hold:

$$d\omega^i = \frac{1}{2}C^i_{jk}[\omega^j\omega^k] + a^i_{j\rho}[\omega^j\theta^\rho] \quad (\rho, r = n+1, \dots, r); \tag{3}$$

$$d\theta^\rho = \frac{1}{2}C^\rho_{\lambda\mu}[\theta^\lambda\theta^\mu] + C^\rho_{l\tau}[\omega^l\theta^\tau] + \frac{1}{2}C^\rho_{lm}[\omega^l\omega^m], \tag{4}$$

where ω^i, θ^ρ are independent forms among ω^i, ω^j_i; $C^i_{jk}, C^\rho_{\lambda\mu}, C^\rho_{l\tau}, a^i_{j\rho}, C^\rho_{lm}$ are the structural constants of the group G. They satisfy the relations following from the Jacobi identities:

$$a^i_{l\rho}a^l_{j\sigma} - a^i_{l\sigma}a^l_{j\rho} = C^\tau_{\sigma\rho}a^i_{j\tau}, \tag{5}$$

$$a^k_{l\tau}C^\tau_{mp} - a^k_{m\tau}C^\tau_{l\rho} = C^i_{lm}a^k_{i\rho} + C^k_{il}a^i_{m\rho} - C^k_{im}a^i_{l\rho}, \tag{6}$$

$$a^k_{l\tau}C^\tau_{mn} + a^k_{m\tau}C^\tau_{nl} + a^k_{n\tau}C^\tau_{lm} = C^i_{lm}C^k_{in} + C^i_{mn}C^k_{il} + C^i_{nl}C^k_{im}. \tag{7}$$

and so on. Here $A_\rho = \|a^i_{j\rho}\|$ are the matrices of infinitesimal rotations of the isotropy group.

If the frames are chosen so that $e_1, \dots, e_q$ lie in E_0, and $e_{q+1}, \dots, e_n$ in E_1, then

$$a^{i_0}_{j\rho} = a^j_{i_0\rho} = 0. \tag{8}$$

As a consequence of this, from equations (6) we obtain the following results:

1) $C^{i_1}_{j_1k_0} = 0, \quad C^{i_0}_{j_0k_1} = 0.$

2) The matrices $C^{i_0} = \|C^{i_0}_{i_1k_1}\|$ commute with the matrices A_ρ for all i_0 and ρ.

3) Each of the operators

$$X_{k_0} = C^{i_1}_{j_1k_0}x^{j_1}\frac{\partial}{\partial x^{i_1}}$$

generates a linear normalizer of the group H in E_1.

3. In what follows the following algebraic propositions will be used.

Lemma 1. Let H be an irreducible orthogonal group acting in the m-dimensional space E_m; let H' be the set of all rotations in E_m commuting with each rotation from H, which evidently also forms a group. Then only three cases are possible: 1) H' contains no rotation; 2) H' is a one-parameter group; 3) H' is a three-parameter group H'_3.

Lemma 2 (^{3}). If there exists a rotation g commuting with the group H, then $m = 2s$, and in a suitable coordinate system the rotation g is represented by the matrix

$$I_1 = \left\| \begin{matrix} 0 & E_s \\ -E_s & 0 \end{matrix} \right\| \quad (E_s \text{ is the identity matrix of order } s) \tag{9}$$

in the orthogonal algebra D_s.

Lemma 3. If $H' = H_3'$, then $m = 4s$, and in a special coordinate system H_3' corresponds to a subalgebra of the orthogonal algebra D_{2s} with basis matrices

$$I_1 = \left\| \begin{matrix} 0 & E_{2s} \\ -E_{2s} & 0 \end{matrix} \right\|, \qquad I_2 = \left\| \begin{matrix} A & B \\ B & -A \end{matrix} \right\|, \qquad I_3 = \left\| \begin{matrix} B & -A \\ -A & -B \end{matrix} \right\|, \tag{10}$$

where A, B are skew-symmetric commuting matrices of order $2s$ satisfying the equality

$$A^2 + B^2 = -E_{2s}.$$

Let us note that I_1, I_2, I_3 and E_{4s} constitute a basis of the matrix representation of the field of quaternions.

Lemma 4. If g is a one-dimensional linear normalizer of an irreducible orthogonal group H in E_m, then g is represented by the matrix $\lambda E + A + B$, where E is the identity matrix, $A = \|a^\alpha_\beta\|$ belongs to the Lie algebra of the group H, and $B = \|b^\alpha_\beta\|$ belongs to the Lie algebra of the group H'.

4. If the isotropy group H admits no rotations commuting with it, then, according to the theorem cited above, the space V_n is semireducible with metric (1). In the case $\sigma \neq$ const, this theorem can be somewhat sharpened; namely, $g_{i_1 j_1}(x^{k_1})\,dx^{i_1} dx^{j_1}$ in (1) must be Euclidean. Therefore one can choose the coordinate system so that

$$ds^2 = g_{i_0 j_0}(x^{k_0})\,dx^{i_0} dx^{j_0} + e^{-2x^1}\left[(dx^{q+1})^2 + \cdots + (dx^n)^2\right]. \tag{11}$$

Moreover, we also obtain the following theorem.

Theorem 1. A homogeneous Riemannian space V_n ($ds^2 > 0$) does not admit a group of similarities broader than the group of motions, except in the case when V_n is Euclidean.

5. Let us now consider the case when the isotropy group H admits only a one-parameter group of permutable rotations. Then equations (2) can be written in the form

$$d\omega^1 = \frac{1}{2} C^1_{j_0 k_0} [\omega^{j_0} \omega^{k_0}] + C \sum_\alpha [\omega^\alpha, \omega^{\alpha+s}] \qquad \left(s = \frac{n-q}{2},\ \alpha = q+1, \dots, q+s \right);$$

$$d\omega^a = \frac{1}{2} C^a_{j_0 k_0} [\omega^{j_0} \omega^{k_0}] \qquad (a = 2, \dots, q), \tag{12}$$

$$d\omega^{i_1} = \frac{1}{2} C^{i_1}_{j_1 k_1} [\omega^{j_1} \omega^{k_1}] + C_{m_0} [\omega^{i_1} \omega^{m_0}] + b^{i_1}_{l_1 m_0} [\omega^{l_1} \omega^{m_0}] + d^{i_1}_{j_1 \rho} [\omega^{j_1} \theta^\rho],$$

where the matrices $B_{m_0} = \| b^{i_1}_{l_1 m_0} \|$ are permutable with the matrices A_ρ for all m_0 and ρ; C is a constant.

a) If $C = 0$, then we return to case (11), or to the reducible metric (1) with $\sigma = \mathrm{const}$.

b) If $C \neq 0$, and all $C_{m_0} = 0$, then in some coordinate system the metric of the space is reduced to the form

$$ds^2 = ds_0^2(x^a) + \left[dx^1 + \omega(x^a, dx^a) + \frac{C}{2} \sum_\alpha (x^\alpha dx^{s+\alpha} - x^{s+\alpha} dx^\alpha) \right]^2 + ds_1^2(x^{i_1}), \tag{13}$$

where $ds_0^2(x^a)$ is a $(q-1)$-dimensional metric admitting a simply transitive group of motions $G^{(1)}$; the forms $\overline{\omega}^1 = dx^1 + \omega(x^a, dx^a)\omega^a$ are invariant forms of the group of motions G_q, consisting of $G^{(1)}$ and a one-dimensional center; $ds_1^2(x^{i_1})$ is an $(n-q)$-dimensional metric with a transitive group of motions $G^{(2)}$, leaving the exterior form $\sum_\alpha [dx^\alpha dx^{\alpha+s}]$ invariant.

c) If $C \neq 0$ and not all $C_{m_0} = 0$, then

$$ds^2 = ds_0^2(x^a) + e^{-2f(x^a)} \left[dx^1 + \omega(x^a, dx^a) + \frac{1}{2} C \sum_\alpha (x^\alpha dx^{\alpha+s} - x^{\alpha+s} dx^\alpha) \right]^2 + $$

$$+ e^{-2f(x^a)} \left[(dx^{q+1})^2 + \dots + (dx^n)^2 \right], \tag{14}$$

where ds_0^2 is a $(q-1)$-dimensional metric admitting a simply transitive group $G^{(1)}$; $df = C_a \omega^a$; the forms $\overline{\omega}^1 = e^{-2f(x^a)} [dx^1 + \omega(x^a, dx^a)]$, ω^a are invariant forms of a certain group G_q, which has a $(q-1)$-dimensional normal divisor with a continuous center; $G^{(1)}$ is a subgroup of G_q.

Let us note that in cases b) and c) $x^{i_1} = \mathrm{const}$ are totally geodesic and imprimitivity submanifolds; however, the entire metric ds^2 is not semireducible, since the field of planes E_1 is nonholonomic. The geodesic lines $x^1 = t$ form a system of imprimitivity of the group of motions and themselves are trajectories of a screw displacement.

6. We pass to a more complicated case, when H admits a three-parameter group of rotations permutable with itself. We denote by $\Omega^1, \Omega^2, \Omega^3$ the exterior forms of the second order corresponding to the skew-symmetric matrices I_1, I_2, I_3. Then equations (3) have the form

$$d\omega^{i_0} = \frac{1}{2} C^{i_0}_{j_0 k_0}[\omega^{j_0}\omega^{k_0}] + C^{i_0}_p \Omega^p; \tag{15}$$

$$d\omega^{i_1} = \frac{1}{2} C^{i_1}_{j_1 k_1}[\omega^{j_1}\omega^{k_1}] + C_{m_0}[\omega^{i_1}\omega^{m_0}] + b^{i_1}_{j_1 m_0}[\omega^{j_1}\omega^{m_0}] + d^{i_1}_{j_1 \rho}[\omega^{j_1}\theta^{\rho}]. \tag{16}$$

Here the following cases are possible:

a) All $C^{i_0}_p = 0$; then the results of item 4 hold.

b) All q forms $C^{i_0}_p \Omega^p$ are proportional; then the results of item 5 are obtained.

c) Among the forms $C^{i_0}_p \Omega^p$, 2 forms are dependent.

d) The forms $C^{i_0}_{p\Omega} \Omega^p$ are expressed in terms of 3 independent forms from among them.

Let us first consider case d). In this case the metric of the space has one of the following two forms:

$$ds^2 = ds_0^2(x^a) + g_{pt}\omega^p\omega^t + ds_1^2(x^{i_1}) \tag{17}$$

$$(a = 4, 5, \dots, q; \quad p, t, u = 1, 2, 3);$$

$$ds^2 = ds_0^2(x^a) + g_{pt}\omega^p\omega^t + e^{-2f(x^a)}[(dx^{q+1})^2 + \dots + (dx^n)^2], \tag{18}$$

where $ds_0^2 = \sum_a (\omega^a)^2$ is a $(q-3)$-dimensional metric admitting a simply transitive group $G^{(1)}$;

$$\omega^p = \varphi^p_t(x^a)[\bar{\omega}^t(x^u, dx^u) + \pi^t(x^a, dx^a) + \sigma^t(x^{i_1}, dx^{i_1})]; \tag{19}$$

$\varphi^p_u(x^a)$ are determined from the completely integrable system

$$d\varphi^p_t = -C^p_{ua}\varphi^u_t\omega^a; \tag{20}$$

g_{pt} are constants corresponding to a symmetric positive matrix of order 3. Moreover,

1) $\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3$ satisfy the relations

$$d\bar{\omega}^1 = \alpha[\bar{\omega}^2\bar{\omega}^3], \quad d\bar{\omega}^2 = \beta[\bar{\omega}^3\bar{\omega}^1], \quad d\bar{\omega}^3 = \gamma[\bar{\omega}^1\bar{\omega}^2], \tag{21}$$

where α, β, γ are constants, equal to zero in case (18);

2) the forms $\varphi^p_t(x^a)[\bar{\omega}^t + \pi^t]$, ω^a determine the group G_q, which contains a $(q-1)$-dimensional normal divisor in case (18);

3) the forms σ^t satisfy the relations

$$\Omega^p = \varphi^p_t d\sigma^t; \tag{22}$$

4) $ds^2_1(x^{i_1})$ in (17) is an $(n-q)$-dimensional metric admitting a group of motions with isotropy group H, which leaves invariant the exterior forms $d\sigma^t$.

We note that the space has ∞^{n-3} imprimitive three-dimensional completely geodesic submanifolds V_3: $x^a = \text{const}$, $x^i = \text{const}$, each geodesic line of which is the trajectory of a displacement. In case (18) each V_3 has a Euclidean metric.

In case c) the results (17) and (18) are preserved, only $p, t = 1, 2$, $a = 3, \dots, q$, and $d\bar{\omega}^1 = d\bar{\omega}^2 = 0$.

We formulate the results obtained in the following theorem.

Theorem 2. *If the isotropy group of a homogeneous Riemannian space V_n leaves fixed all directions in the plane E_0 and acts irreducibly on the orthogonal complement E_1 to E_0, then the metric of the space has the form (1) (in it $\sigma(x^{k_0}) = 1$), (11), (13), (14), (17), or (18).*

We note that in cases (11), (14), (18) the metric is determined by specifying the group G_q with the properties indicated above, and in cases (1), (13), (17) by the group G_q and an $(n-q)$-dimensional metric admitting a transitive group of motions with isotropy group H, which leaves invariant one or three exterior forms of the second order in cases (13) and (17), respectively.

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Note: Figure translations are in progress. See original paper for figures.

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