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Abstract

Full Text

Physics

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On the Theory of Radiation Diffusion in a Semi-Infinite Medium

(Presented by Academician V. A. Ambartsumian on February 6, 1958)

V. V. Sobolev has shown ⁽¹⁾ that the solution of any problem on the luminosity of a plane layer of infinitely large optical thickness with a spherical scattering indicatrix, in the presence of radiation sources whose power depends only on the optical depth τ , is reduced to finding the function $\Phi(\tau)$, defined by the equation

$$\Phi(\tau) = K(\tau) + \int_0^\tau K(\tau - \tau')\Phi(\tau') d\tau', \quad (1)$$

where

$$K(\tau) = \frac{\lambda}{2} \int_0^1 e^{-\tau/\eta} \varphi(\eta) \frac{d\eta}{\eta}; \quad (2)$$

λ is the ratio of the scattering coefficient to the attenuation coefficient; $\varphi(\eta)$ is a function introduced by V. A. Ambartsumian ⁽²⁾ and defined by the equation

$$\varphi(\eta) = 1 + \frac{\lambda}{2} \eta \varphi(\eta) \int_0^1 \frac{\varphi(\zeta)}{\eta + \zeta} d\zeta. \quad (3)$$

In the present note exact analytic solutions of equation (1) are found, and some consequences from them are obtained.

Applying the one-sided Laplace transform to equation (1), we obtain

$$\bar{\Phi}(p) = \frac{\bar{K}(p)}{1 - \bar{K}(p)}, \quad (4)$$

where

$$\bar{\Phi}(p) = \int_0^\infty \Phi(\tau) e^{-p\tau} d\tau, \quad \bar{K}(p) = \frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)}{1 + p\eta} d\eta. \quad (5)$$

Using the well-known Mellin–Riemann inversion formula

$$\Phi(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{\Phi}(p) e^{p\tau} dp \quad (6)$$

and carrying out the integration along the imaginary axis ($c = 0$), we find

$$\Phi(\tau) = \frac{1}{\pi} \int_0^\infty [A(x) \cos \tau x + B(x) \sin \tau x] dx, \quad (7)$$

where

$$A(x) = \frac{\lambda \frac{\operatorname{arc\,tg} x}{x} - a(x)}{1 - \lambda \frac{\operatorname{arc\,tg} x}{x}}, \quad B(x) = -\frac{xb(x)}{1 - \lambda \frac{\operatorname{arc\,tg} x}{x}},$$

$$a(x) = \frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)}{1 + \eta^2 x^2} d\eta, \quad b(x) = \frac{\lambda}{2} \int_0^1 \frac{\eta \varphi(\eta)}{1 + \eta^2 x^2} d\eta.$$

We note that $a(x)$ and $b(x)$ are connected by the relation

$$x^2 b^2(x) + a^2(x) - 2a(x) + \lambda \frac{\operatorname{arc\,tg} x}{x} = 0.$$

However, the use of formula (7) for computations is inconvenient.

Another way of obtaining the transform $\bar{\Phi}(p)$ can be indicated. The point is that $\bar{\Phi}(p)$ has at the point $p = -k$ a pole of the first order, and the quantity k is connected with λ by the equation

$$\frac{\lambda}{2k} \operatorname{lg} \frac{1+k}{1-k} = 1.$$

Moreover, the point $p = -1$ is a branch point for $\bar{\Phi}(p)$. Taking into account the presence of such singular points and using the method of contour integration, we obtain

$$\Phi(\tau) = \frac{e^{-k\tau}}{\frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)\eta}{(1-k\eta)^2} d\eta} + 2\lambda \int_1^\infty \frac{xe^{-\tau x}}{\pi^2 \lambda^2 + (2x + \lambda \operatorname{lg} \frac{x-1}{x+1})^2} \frac{dx}{\varphi\left(\frac{1}{x}\right)}. \quad (8)$$

For $\lambda = 1$, from (8) we find

$$\Phi(\tau) = \sqrt{3} + 2 \int_1^{\infty} \frac{x e^{-\tau x}}{\pi^2 + \left(2x + \lg \frac{x-1}{x+1}\right)^2} \frac{dx}{\varphi\left(\frac{1}{x}\right)}. \quad (9)$$

As shown in (1), the solution of some problems reduces to determining the function

$$\Psi(\tau) = 1 + \int_0^{\tau} \Phi(\tau) d\tau.$$

For $\Psi(\tau)$ it is easy to find

$$\begin{aligned} \Psi(\tau) = & \frac{1}{\sqrt{1-\lambda}} - \frac{e^{-k\tau}}{k \frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)\eta}{(1-k\eta)^2} d\eta} - \\ & - 2\lambda \int_1^{\infty} \frac{e^{-\tau x}}{\pi^2 \lambda^2 + \left(2x + \lambda \lg \frac{x-1}{x+1}\right)^2} \frac{dx}{\varphi\left(\frac{1}{x}\right)}, \end{aligned} \quad (10)$$

and for $\lambda = 1$ we have

$$\Psi(\tau) = 1 + \tau\sqrt{3} + 2 \int_1^{\infty} \frac{1 - e^{-\tau x}}{\pi^2 + \left(2x + \lg \frac{x-1}{x+1}\right)^2} \frac{dx}{\varphi\left(\frac{1}{x}\right)}. \quad (11)$$

We note the relation

$$\frac{1}{\frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)\eta}{(1-k\eta)^2} d\eta} = \frac{k}{\frac{\lambda}{1-k^2} - 1} \left(1 - \frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)}{1+k\eta} d\eta\right),$$

which may prove useful in carrying out computations by formulas (8) and (10).

It follows from formula (8) that for $\tau \gg 1$ the integral term may be neglected and, consequently, the nonintegral term is an exact asymptotic expression for $\Phi(\tau)$ for large τ . For $\Psi(\tau)$, from (10), for $\tau \gg 1$ we find the value $\frac{1}{\sqrt{1-\lambda}}$.

If pure scattering of radiation occurs in the medium ($\lambda = 1$) and the sources of radiation are located at infinitely great depth, then the solution of the problem is usually represented in terms of the function $q(\tau)$, which, as found by V. V. Sobolev¹, is related to $\Psi(\tau)$ by

$$q(\tau) = \frac{\Psi(\tau)}{\sqrt{3}} - \tau. \tag{12}$$

Using (11) and (12), for $q(\tau)$ we have

$$q(\tau) = \frac{1}{\sqrt{3}} \left[1 + 2 \int_1^\infty \frac{1 - e^{-\tau x}}{\pi^2 + (2x + \lg \frac{x-1}{x+1})^2} \frac{dx}{\varphi\left(\frac{1}{x}\right)} \right]. \tag{13}$$

The function $q(\tau)$ in the form (13) was obtained by another method by Mark ³. Here we have found (13) from the general solution (10), putting $\lambda = 1$ in it.

Table 1

τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ	τ																																				
0,00	0,01	0,02	0,03	0,05	0,10	0,20	0,30	0,40	0,60	0,80	1,00	1,50	2,00	2,50	3,00	∞	$\Phi(\tau)$	$\Psi(\tau)$	$q(\tau)$																																		
∞	2,89	2,77	2,66	2,49	2,24	2,02	1,93	1,87	1,82	1,79	1,77	1,75	1,74	1,73	1,73	1,73	1,00	1,04	1,06	1,09	1,14	1,26	1,47	1,67	1,86	2,23	2,59	2,94	3,82	4,69	5,56	6,43	∞	0,57	0,58	0,59	0,60	0,61	0,62	0,63	0,65	0,66	0,67	0,68	0,69	0,69	0,70	0,70	0,70	0,71	0,71	0,71	0,71

In conclusion we give Table 1 of the functions $\Phi(\tau)$, $\Psi(\tau)$, and $q(\tau)$ for $\lambda = 1$, calculated by formulas (9), (11), and (13).

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- ³ C. Mark, Phys. Rev., **72**, No. 7 (1947).

Note: Figure translations are in progress. See original paper for figures.

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