On an Equation of Infinite Order with Polynomial Coefficients

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Abstract

Full Text

Mathematics

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Until recently, the differential equation of infinite order with constant coefficients

$$a_0 y + a_1 y' + a_2 y'' + \cdots = f(x) \tag{1}$$

was studied under the assumption that the characteristic function $a_0 + a_1 x + a_2 x^2 + \cdots$ is analytic in some disk centered at the origin. In the work ([1]), the case of rapidly growing coefficients, when the characteristic function does not exist, was first considered.

The equation with polynomial coefficients

$$P_0(x) y + P_1(x) y' + P_2(x) y'' + \cdots = f(x) \tag{2}$$

has been investigated by various authors. In recent years it has been studied in works ([2–4]) under two obligatory assumptions: a) the degrees of the polynomials

$$P_i = \sum_{k=0}^{p_i} a_i^k x^k$$

are bounded by one and the same number $p_i \le p$; b) the characteristic functions

$$\omega_k(x) = \sum_{i=0}^{\infty} a_i^k x^i, \quad k = 0, 1, \dots, p,$$

are analytic in some disk centered at the origin.

In the present note we establish the existence and uniqueness (in a certain class of analytic functions) of a solution of equation (2) without these assumptions. In addition, a method is given for the approximate solution of equation (2), and an error bound is indicated when the exact solution is replaced by an approximate one. The method of proof is based on the theory of infinite systems of linear algebraic equations ([5]).

§ 1. **Definition.** We shall call equation (2) **regular** if:
1) $P_0(x) \equiv a_0 \neq 0$; 2) $P_i(x)$ is a polynomial of degree not exceeding $i-1$, $i = 1, 2, \ldots$; 3) $f(x)$ is an entire function. We shall regard the entire function $y(x)$ as a solution of a regular equation in a domain Q containing the origin if the series $\sum_{k=0}^{\infty} P_k(x) y^{(k)}$ converges uniformly inside Q and its sum is equal to $f(x)$. A regular equation can always be reduced to the following canonical form:

$$y + \sum_{k=1}^{\infty} P_k(x) y^{(k)} = f(x), \qquad P_k(x) = \sum_{s=0}^{k-1} a_k^s x^s. \tag{3}$$

Lemma. Let $\{A_k^s\}$, $s = 0, 1, \ldots, k-1$, $k = 1, 2, \ldots$, be any sequence majorizing $\{a_k^s\}$. Then one can specify a sequence of positive numbers $A(A_0, A_1, \ldots)$ that satisfies the conditions:

$$\overline{\lim_{k\to\infty}} \frac{1}{A_k} \sum_{m=1}^{k-1} m! A_m \sum_{s=0}^{m} \frac{A_{k-m+s}^s}{(m-s)!} < 1, \qquad \overline{\lim_{k\to\infty}} \frac{\sum_{m=0}^{k-1} A_m}{A_k} < \infty, \qquad \overline{\lim_{k\to\infty}} \frac{A_k^0}{A_k} < \infty. \tag{4}$$

It is convenient to consider equation (3) in a special subclass of entire functions. Namely, let $A(A_0, A_1, \ldots)$ be any sequence of positive numbers. Denote by K_A the set of entire functions such that

$$\sum_{k=0}^{\infty} |f^{(k)}(0)| A_k < \infty.$$

Theorem 1. Let $A(A_0, A_1, \ldots)$ be an arbitrary sequence of positive numbers satisfying conditions (4). Then, for any right-hand side $f(x)$ from K_A, equation (3) has a unique solution in the class K_A. This solution satisfies the equation in any bounded domain. If $y(x)$ is a solution of equation (3) from K_A, then

$$\sum_{k=0}^{\infty} |y^{(k)}(0)| A_k \leq D \sum_{k=0}^{\infty} |f^{(k)}(0)| A_k,$$

where D is a constant independent of $f(x)$, $y(x)$.

For an approximate solution of equation (3), one may use the method of "truncation." Namely, the ordinary equation

$$y + P_1(x) y' + P_2(x) y'' + \cdots + P_n(x) y^{(n)} = f_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \tag{5}$$

has one and only one polynomial solution y_n (the degree of y_n is equal to the degree of f_n).

Theorem 2. If the numbers $\{A_k\}$ satisfy conditions (4), $f(x) \in K_A$, and $y(x)$ is the corresponding solution from K_A, then $y_n(x) \to y(x)$ uniformly in every bounded domain. Moreover, in any such domain the expression

$$\sum_{k=0}^{n} A_k |y^{(k)}(x) - y_n^{(k)}(x)|$$

tends uniformly to zero. The order of approximation is characterized by the inequality

$$\sum_{k=0}^{n} A_k |y^{(k)}(0) - y_n^{(k)}(0)| \leq d \sum_{k=n+1}^{\infty} A_k |f^{(k)}(0)|,$$

where d is a constant of the same nature as D (in Theorem 1).

Theorem 3. Under the conditions of Theorem 2, there is the representation

$$y(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z_k(x), \tag{6}$$

where $z_n(x)$ is the polynomial solution of the equation

$$y + \sum_{k=1}^{n} P_k(x) y^{(k)} = x^n, \qquad n = 0, 1, 2, \ldots$$

The series (6) converges uniformly in every bounded domain, and

$$\sup_{|x| \leq R} \left| y(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z_k(x) \right| \leq B_R \sum_{k=n+1}^{\infty} A_k |f^{(k)}(0)|.$$

§ 2. Let us consider some consequences of Theorems 1-3.

1. Denote by M_ε the subclass of entire functions $g(x)$ of zero order satisfying the condition

$$\|g\|_\varepsilon = \sum_{k=0}^{\infty} |g^{(k)}(0)| (k!)^{k\varepsilon} < \infty, \quad \varepsilon > 0.$$

Theorem 4. Suppose that each polynomial $P_m(x)$ has a majorant of the form $AR^m e^{cx}$, i.e. $|a_m^s| \leq AR^m c^s / s!$ for all m and s. Then, for any right-hand side $f(x)$ from M_ε, there exists a unique solution of equation (3) in M_ε. Moreover,

$$\|y\|_\varepsilon \leq D_1 \|f\|_\varepsilon,$$

$$\sum_{k=0}^{n}(k!)^{k\varepsilon}\left|y^{(k)}(0)-y_n^{(k)}(0)\right| \le D_2 \sum_{k=n+1}^{\infty}(k!)^{k\varepsilon}|f^{(k)}(0)|,$$

where the constants D_i do not depend on f or y.

2. Suppose that all polynomials P_i, beginning with the $(p+1)$-st, are of degree no higher than p $(p \ge 0)$. Then equation (3) takes the form

$$y+\sum_{k=1}^{p} y^{(k)} \sum_{s=0}^{k-1} a_k^s x^s + \sum_{k=p+1}^{\infty} y^{(k)} \sum_{s=0}^{p} a_k^s x^s = f(x). \tag{7}$$

Denote by C_μ^β the class of entire functions of growth no greater than

$$\left[\frac{1}{1+\mu};\ (1+\mu)\beta^{\frac{1}{1+\mu}}\right], \quad \mu>0,\ \beta>0.$$

Theorem 5. Let the coefficients a_n^s of equation (7) satisfy the condition

$$\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n^s|(n!)^{-\mu}} < \beta, \quad s=0,\dots,p. \tag{8}$$

If $\mu > p$, then: a) for any right-hand side $f(x)$ from the class C_μ^β, equation (7) has a unique solution in the same class; b) whatever $R<\infty$ may be,

$$\overline{\lim}_{n\to\infty} \sqrt[n]{\max|y(x)-y_n(x)|} \le q_f < 1,$$

and for a function f of order less than $1+\mu$, $q_f = 0$.

In connection with Theorem 5 it is interesting to note the result of Sikkema (*), who showed that condition (8) is necessary and sufficient for the series

$$\sum_{k=0}^{\infty} P_k(x) y^{(k)}, \quad \text{where } P_k(x)=\sum_{s=0}^{p} a_k^s x^s,$$

to converge at every finite value of x for any entire function in the class C_μ^β. Thus, in the study of equation (7) in the class C_μ^β, condition (8) turns out to be completely natural.

Part a) of Theorem 5 in the special case $p=0$, i.e. for an equation with constant coefficients, was obtained earlier ([1]).

§ 3. In the case where the degrees of some of the polynomials P_k exceed $k-1$, a number of special features may arise. A solution of equation (2) exists, generally speaking, not for every right-hand side $f(x)$; the uniqueness property is also violated—the solution depends on one or several arbitrary constants.

We shall confine ourselves here to a summary of results for the equation with linear coefficients

$$\sum_{k=0}^{\infty}(a_k + xd_k)y^{(k)} = f(x). \tag{9}$$

Without loss of generality one may assume that at least one of the coefficients a_0, d_0 is nonzero. Then only the following cases are possible:

1. $d_0 \neq 0$. In order that a solution of equation (9) exist, it is necessary that $f(t)$ satisfy a certain condition (the condition is imposed only on $f(0)$). For any function $f(t)$ from K_A satisfying this condition, where $A(A_0, A_1, ...)$ is an arbitrary sequence for which the properties (4) hold, there exists a unique solution in K_A. The method of approximate solution indicated above carries over to this case with minor changes and gives an error estimate.

2. $d_0 = 0$; then $a_0 \neq 0$, and the character of equation (9) is determined by the following pair of coefficients. Namely:

2a) If $d_1 = 0$, then the equation is regular.

2b) $d_1 \neq -a_0/n,\ n = 1, 2, ...$; the solution exists for any $f(x) \in K_A$ and is unique in the same class; all results are exactly the same as for a regular equation.

2c) $d_0 = -a_0/p,\ p$ a positive integer. In this case, for the existence of a solution it is necessary that the right-hand side $f(x)$ satisfy a certain condition (only the quantity $f^{(p)}(0)$ enters into this condition). If this condition is fulfilled for $f(x)$ and if $f(x) \in K_A$, then the solution exists in the same class and depends on one arbitrary constant.

The cases considered may be combined in the following general result (as before, it is assumed that $|a_0|^2 + |d_0|^2 > 0$):

Theorem 6. Let $A(A_0, A_1, ...)$ be an arbitrary sequence satisfying conditions (4).

In order that equation (9) have a unique solution in K_A for any right-hand side $f(x)$ from K_A, it is necessary and sufficient that the first-order equation

$$(a_0 + xd_0)y + (a_1 + xd_1)y' = x^n$$

have, for all $n,\ n = 0, 1, 2, ...$, a polynomial solution of exactly degree n. If this condition is fulfilled, then as an approximate solution one may take the polynomial solution y_n of the equation

$$(a_0 + d_0x)y + (a_1 + d_1x)y' + \dots + (a_n + d_{nx})y^{(n)} = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

In this case $y_n(x) \to y(x)$ uniformly in any bounded domain.

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Note: Figure translations are in progress. See original paper for figures.