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Abstract

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MATHEMATICS

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DETERMINATION OF A CONVEX SURFACE BY A GIVEN FUNCTION OF ITS PRINCIPAL CURVATURES

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1. Let a convex surface F be given by the equation $z = z(x, y)$ $(z(x, y) \in C^2)$ in some convex domain D in the x, y-plane, and let K_1, K_2 be the principal curvatures of the surface F. In this paper we shall consider questions of existence of a convex surface F, if at each point $(x, y) \in D$ the principal curvatures of the surface F are connected by the relation

$$f(x, y, z, p, q)K_1K_2 - \varphi(x, y, p, q)(K_1 + K_2) = \psi(x, y) \tag{1}$$

or by the relation

$$f(x, y, z, p, q)K_1K_2 - \varphi(x, y, p, q)\sqrt{f(x, y, z, p, q)}(K_1 + K_2) = \psi(x, y), \tag{2}$$

where in both cases the functions $f(x, y, z, p, q)$ and $\varphi(x, y, p, q)$ are continuous in x, y in D and for all values of the variables z, p, q, and $f(x, y, z, p, q) > 0$, $\varphi(x, y, p, q) > 0$. The function $\psi(x, y)$ is assumed summable in D.

Relations (1) and (2) may be regarded as second-order differential equations with respect to the function $z(x, y)$, if, instead of K_1, K_2, their expressions through the first and second derivatives of the function $z(x, y)$ are substituted. The differential equations (1) and (2) will be of elliptic type if, on their solutions, respectively the relations

$$f(x, y, z, p, q)\psi(x, y) + \varphi^2(x, y, p, q) > 0; \tag{3}$$

$$\psi(x, y) + \varphi^2(x, y, p, q) > 0. \tag{4}$$

Introduce the operators, defined on the class $C^2(D)$:

$$\Phi_1(z) = f(x,y,z,p,q)K_1K_2 - \varphi(x,y,p,q)(K_1+K_2); \tag{5}$$

$$\Phi_2(z) = f(x,y,z,p,q)K_1K_2 - \varphi(x,y,p,q)\sqrt{f(x,y,z,p,q)}(K_1+K_2), \tag{6}$$

In ([1]) it was proved that the ellipticity of the expressions $\Phi_1(z)$ ($\Phi_2(z)$) on solutions of equations (1) ((2)) is equivalent to the fact that $\partial\Phi_1/\partial K_i > 0$ $(i=1,2)$ or $\partial\Phi_1/\partial K_i < 0$ $(i=1,2)$ ($\partial\Phi_2/\partial K_i > 0$ $(i=1,2)$ or $\partial\Phi_2/\partial K_i < 0$ $(i=1,2)$). For definiteness, below we shall restrict ourselves to consideration of the case when $\partial\Phi_1/\partial K_i > 0$ or $\partial\Phi_2/\partial K_i > 0$. In this case, using the conditions imposed on the functions f and φ, from the conditions $\partial\Phi_1/\partial K_i > 0$ $(i=1,2)$ ($\partial\Phi_2/\partial K_i < 0$ $(i=1,2)$) we obtain that the solutions of equations (1) and (2) will be convex functions whose convexity is turned toward $z<0$.

The Dirichlet problem for equations (1) and (2) will have a unique solution in the class C^2 if $f(x,y,z,p,q)$ and $\varphi(x,y,p,q)$ are continuously differentiable with respect to z,p,q and $f_z(x,y,z,p,q) \le 0$.

2. Let us introduce the concept of generalized solutions of equations (1) and (2). For this purpose we extend the operators $\Phi_1(z)$ and $\Phi_2(z)$, defined on C^2, to the class

of all convex functions. Let us consider the extension $\Phi_1(z)$, since for $\Phi_2(z)$ the construction is analogous. Let first the convex function $z(x,y) \in C^2$. Integrating (5) over a variable Borel set e of the domain D:

$$\iint_e \Phi_1(z)\,dx\,dy = \iint_e f(x,y,z,p,q)K_1K_2\,dx\,dy - \iint_e \varphi(x,y,p,q)(K_1+K_2)\,dx\,dy.$$

The set function

$$\omega(z,f,e) = \iint_e f(x,y,z,p,q)K_1K_2\,dx\,dy,$$

constructed on the convex surface F given by the equation $z=z(x,y)$, can, by means of the normal mapping μ onto the plane (p,q) (see (2)), be transformed to the form

$$\omega(z,f,e) = \iint_{\mu(e)} \frac{f(x(p,q),y(p,q),z(x(p,q),y(p,q)),p,q)}{(1+p^2+q^2)^2}\,dp\,dq.$$

In papers ([2–4]) it is proved that if $f(x,y,z,p,q)$ is a continuous function of its arguments, then the set function $\omega(z,f,e)$ can be extended to an arbitrary convex surface as a completely additive nonnegative function of Borel sets of the domain D. The properties of these set functions are studied there as well. The set function

$$h(z,\varphi,e)=\iint_e \varphi(x,y,p,q)(K_1+K_2)\,dx\,dy$$

for every convex function $z\in C^2$ can be brought to the form

$$h(z,\varphi,e)=2\iint_{\nu(e)} \frac{\varphi(x(\xi,\eta),y(\xi,\eta),p(\xi,\eta),q(\xi,\eta))}{\sqrt{1+p^2(\xi,\eta)+q^2(\xi,\eta)}}\,\widetilde{H}(d\tilde{e}). \tag{7}$$

The meaning of the parameters ξ,η and of the set function $\widetilde{H}(\tilde{e})$ is as follows. Consider the convex surface F_λ, parallel to the surface F and constructed along the outer normals to F with length λ. F_λ is given by the vector function

$$\mathbf{r}(x,y,\lambda)=\left(x-\frac{\lambda p}{\sqrt{1+p^2+q^2}}\right)\mathbf{i}+\left(y-\frac{\lambda q}{\sqrt{1+p^2+q^2}}\right)\mathbf{j}+\left(z(x,y)+\frac{\lambda}{\sqrt{1+p^2+q^2}}\right)\mathbf{k}.$$

Put

$$\xi=x-\frac{p}{\sqrt{1+p^2+q^2}},\qquad \eta=y-\frac{q}{\sqrt{1+p^2+q^2}}.$$

Then the mapping $\nu:\ \xi=\xi(x,y),\ \eta=\eta(x,y)$ is one-to-one. The mapping $\nu^{-1}:\ x=x(\xi,\eta),\ y=y(\xi,\eta)$ makes it possible to determine uniquely

$$p(\xi,\eta)=p(x(\xi,\eta),y(\xi,\eta)),\qquad q=q(x(\xi,\eta),y(\xi,\eta)).$$

The set function

$$H(e)=\frac{1}{2}\iint_e (K_1+K_2)\sqrt{1+p^2+q^2}\,dx\,dy,$$

which is the mean integral curvature of F, is connected with the area on the surface F_λ by the relation

$$\sigma(F_\lambda,e)=\iint_e \sqrt{1+p^2+q^2}\,dx\,dy+$$

$$+2\lambda\iint_e \frac{K_1+K_2}{2}\sqrt{1+p^2+q^2}\,dx\,dy+\lambda^2\iint_e K_1K_2\sqrt{1+p^2+q^2}\,dx\,dy.$$

Hence

$$H(e) = \frac{d}{d\lambda}\sigma(F_\lambda, e)\Big|_{\lambda=0}.$$

The set function $\widetilde{H}(\tilde{e})$ on the ξ, η plane is constructed as follows:

$$\widetilde{H}(\tilde{e}) = \widetilde{H}(\tilde{e} \cap \nu(D)) = H(\nu^{-1}(\tilde{e} \cap \nu(D))).$$

For any convex surface F, given by a function $z(x, y)$, the mapping $\nu^{-1} : x = x(\xi, \eta),\ y = y(\xi, \eta)$ is one-to-one and, with the aid of the formulas

$$\xi = x - \frac{p}{\sqrt{1 + p^2 + q^2}}, \qquad \eta = y - \frac{q}{\sqrt{1 + p^2 + q^2}},$$

which connect ξ, η, x, y with the coefficients p and q of the supporting planes of F at the point $(x(\xi, \eta), y(\xi, \eta))$, $p(\xi, \eta)$, $q(\xi, \eta)$, are determined as single-valued continuous functions of the variables ξ and η. The set functions $H(e)$ and $\widetilde{H}(\tilde{e})$ will be completely additive nonnegative set functions. Therefore, by means of formula (7), the set function $h(z, \varphi, e)$ can be extended to arbitrary convex surfaces F and it can be established that $h(z, \varphi, e)$ is a completely additive nonnegative set function on the Borel subsets of the domain D, if $\varphi(x, y, p, q)$ is continuous in x, y, p, q. Let us consider the completely additive set function

$$\Phi_1(z, e) = \omega(z, f, e) - h(z, \varphi, e).$$

If the surface F is given by a convex function $z(x, y) \in C^2$, then

$$\Phi_1(z, e) = \iint_e \Phi_1(z)\, de.$$

Let $\Psi(e)$ be a completely additive function of bounded variation, given in D. By a solution of the equation

$$\Phi_1(z, e) = \Psi(e) \tag{8}$$

we shall mean a convex function z, with convexity directed toward $z < 0$, such that the indicated equation is satisfied on all Borel subsets of the domain D that are separated from the boundary of D by a positive distance. In particular, if

$$\Psi(e) = \iint_e \varphi(x, y)\, dx\, dy,$$

then the solutions of equation (8) will be called generalized solutions of equation (1).

The notion of generalized solutions of equation (2) is constructed analogously; here it is naturally assumed that $f(x, y, z, p, q)$ and $\varphi(x, y, p, q)$ are continuous functions of their variables.

3. Let now D be a convex domain in the x, y plane, bounded by a closed convex curve Γ, and let L be a certain curve in space that projects one-to-one onto Γ.

We shall next consider functions $f(x, y, z, p, q)$ and $\varphi(x, y, p, q)$ satisfying the following conditions: 1) $f(x, y, z, p, q)$ and $\varphi(x, y, p, q)$ are continuous in all variables: in x, y in D and in z, p, q for all finite values of these variables; 2) $f(x, y, z, p, q)$ has a nonpositive derivative $f_z(x, y, z, p, q)$, continuous in x, y in D and for all finite values of z, p, q.

Let T be the boundary of the convex hull spanned by the curve L, and let T' be the part of it lying under L. T' is a convex developable surface, with convexity directed downward. Then, for any Borel set $e \subset D$, we have

$$\omega(T', f, e) = 0, \qquad h(T', \varphi, e) \geq 0, \qquad h(T', \varphi\sqrt{f}, e) \geq 0.$$

In the domain D we shall now consider completely additive set functions $\Psi_1(e)$ and $\Psi_2(e)$, satisfying respectively the conditions:

$$-h(T', \varphi, e) \leq \Psi_1(e); \qquad -h(T', \varphi\sqrt{f}, e) \leq \Psi_2(e).$$

Let us now consider in D two classes of convex functions W_1 and W_2 such that: 1) for any Borel set $e \subset D$ we have $\Phi_1(u, e) \leq \Psi_1(e)$, if $u \in W_1$, and $\Phi_2(u, e) \leq \Psi_2(e)$, if $u \in W_2$; 2) all functions of both classes W_1, W_2 are convex downward; 3) if u belongs to W_1 or W_2, then in the corresponding class W_i there is no convex surface v such that $u - v = \text{const}$, and moreover the boundary points of the surface u lie strictly below the boundary points of the surface v; 4) all surfaces defined by functions from W_1 or from W_2 lie below the surface T' and have common points with L.

The classes of functions W_1, W_2, obviously, are nonempty. Let $z \in W_1$ or $z \in W_2$. Denote by $v(z)$ the volume of the body bounded by the surfaces $z(x, y)$, T and the cylinder with directrix Γ and generators parallel to the z-axis.

Theorem 1. *If for the classes of convex functions W_i $(i = 1, 2)$ the relation*

$$v_0 = \sup_{z \in W_i} v(z) < +\infty$$

holds, then there exists a function $z_0 \in W_i$ $(i = 1, 2)$ realizing this maximum. For all Borel sets $e \subset D$ at a positive distance from Γ, this function satisfies the relation $\Phi_i(z, e) = \Psi_i(e)$ $(i = 1, 2)$. If $\Psi_i(e)$ is an absolutely continuous set function, then z_0 is a generalized solution, respectively, of equation (1) or (2), and almost everywhere satisfies equation (1) or (2).

If Theorem 1 is considered in the class W_1, then condition (2) on the existence of f_z and the sign of f_z may be omitted. We give sufficient conditions which ensure a finite estimate of v_0 in the classes W_1 and W_2.

Theorem 2. *Let the functions $f(x, y, z, p, q)$ and $\varphi(x, y, p, q)$ satisfy the conditions: 1) there exists a function $f_0(p, q)$, summable over the whole p, q-plane,* *such that* $f(x,y,z,p,q) \le f_0(p,q)$ *for all* $x,y,z,$ *and*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f_0(p,q)}{(1+p^2+q^2)^2}\,dp\,dq < +\infty;$$

2) for all x,y,p,q *we have*

$$\varphi(x,y,p,q) \ge \text{const} > 0.$$

Then for the number v_0 *in the class* W_1 *the inequality* $v_0 < +\infty$ *holds. An analogous assertion holds in the class* W_2*, if condition 2) is replaced by the inequality*

$$\varphi(x,y,p,q)\sqrt{f(x,y,z,p,q)} \ge \text{const} > 0.$$

Theorem 3. Denote by

$$B_1 = \sup_{z\in W_1} h(z,\varphi,D)$$

and by

$$B_2 = \sup_{z\in W_2} h(z,\varphi\sqrt{f},D),$$

and suppose that there exists, summable in every bounded domain of the p,q-plane, a function $f_1(p,q)$ such that for all x,y,z we have $f(x,y,z,p,q) \ge f_1(p,q)$.

Then, if $B_i < +\infty$ $(i=1,2)$ and the inequality

$$B_i + \operatorname*{Var}_D \Psi_i(e) < \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f_1(p,q)}{(1+p^2+q^2)^2}\,dp\,dq \quad (i=1,2)$$

holds, then for v_0 a finite upper estimate can be obtained in the classes W_i $(i=1,2)$.

A finite estimate for the numbers B_1 and B_2 can be obtained if we have, respectively,

$$\varphi(x,y,p,q) \le \frac{A_1}{(1+p^2+q^2)^\alpha} \quad \text{and} \quad \varphi\sqrt{f} \le \frac{A_2}{(1+p^2+q^2)^\alpha},$$

where $\alpha > 2$ and A_1, A_2 are positive constants.

All the results considered are generalized verbatim to the case of n variables for the equations

$$f(x_1,\dots,x_n,z,p_1,\dots,p_n)K_1\cdots K_n - \varphi(x_1,\dots,x_n,p_1,\dots,p_n)(K_1+\dots+K_n) = \psi(x_1,\dots,x_n),$$

$$f(x_1,\dots,x_n,z,p_1,\dots,p_n)K_1\cdots K_n - \varphi(x_1,\dots,x_n,p_1,\dots,p_n)\sqrt{f(x_1,\dots,x_n,z,p_1,\dots,p_n)}\times(K_1+\dots+K_n) = \psi(x_1,\dots$$

If $\varphi \equiv 0$, then from Theorems 3 and 1 follow the existence theorems for the Dirichlet problem, established in works ([2–4]), for the equations

$$\Gamma(z) = \varphi(x_1,\dots,x_n,z,p_1,\dots,p_n),$$

where $\Gamma(z)$ is the Hessian of the function $z(x_1, \dots, x_n)$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.