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Abstract

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ON THE MULTIDIMENSIONAL LIMITING DISTRIBUTION OF THE NUMBERS OF BUSY LINES IN SWITCHES OF THE SECOND STAGE OF A TELEPHONE SYSTEM WITH LOSSES

(Presented by Academician V. S. Kulebakin, January 29, 1958)

1. In the present note it is proved that the limiting distribution of the numbers of busy lines in the switches of the second stage of a two-stage telephone system with losses, operating in the free-search regime, is approximated by the density function of a multidimensional normal distribution, under the condition that the number of switches of the first stage is sufficiently large. It is assumed here that a stationary Poisson stream of calls with constant parameter λ arrives at each of the switches of the first stage, that the holding time of each line has an exponential distribution, and also that the switches of the first stage function independently of one another and independently of the switches of the second stage. The last condition is fulfilled if in each of the switches of the second stage the number of outlets is greater than or equal to the number of inlets.

2. Consider a two-stage scheme consisting of k switches of the first stage and m switches of the second stage. Suppose that each switch of the first stage has N inlets and m outlets, $\min(N, m) = n$, and each of the switches of the second stage has k inlets and l outlets, where $l \geq k$. Let a Poisson stream of calls with parameter λ arrive at the inlets of each of the switches of the first stage, and let the streams arriving at different switches be mutually independent. Each of the incoming calls may, with equal probability, occupy any of the free connecting paths leading to the switches of the second stage. The order in which the free outgoing lines of the switches of the second stage are seized may be arbitrary—random and equiprobable, or sequential. A call arriving at some switch of the first stage receives a loss only in the case when n lines of this switch are busy. The condition $l \geq k$ ensures the absence of losses due to the occupation of

lines of the switches of the second stage, and consequently also the mutual independence of the operation of the switches of the first stage.

Suppose that the holding time of each line has an exponential distribution with mean value 1 and that the duration of occupation of one line does not depend on the duration of occupation of other lines, nor on the incoming calls.

3. Associate with each line connecting switch i of the first stage with switch j of the second stage a number s_{ij}, and assume that $s_{ij} = 1$ if this line is busy at time t, and that $s_{ij} = 0$ if the line is free. Obviously,

$$s_{i\cdot} = \sum_{j=1}^{m} s_{ij} \quad (i = 1, 2, \dots, k)$$

and

$$s_{\cdot j} = \sum_{i=1}^{k} s_{ij} \quad (j = 1, 2, \dots, m)$$

are equal, respectively, to the number of busy lines at time t in switch i of the first stage and in switch j of the second. Thus, the instantaneous state of the two-stage system under consideration

is one of the points in the space Ω of rectangular matrices $\|s_{ij}\|$ $(i = 1, 2, \dots, k;\ j = 1, 2, \dots, m)$, whose elements are equal to 0 or 1, and the sums of the elements by rows and by columns satisfy, respectively, the conditions:

$$0 \leq s_{i\cdot} \leq n, \qquad i = 1, 2, \dots, k; \tag{1}$$

$$0 \leq s_{\cdot j} \leq l, \qquad j = 1, 2, \dots, m. \tag{2}$$

In the case $l \geq k$, condition (2) is satisfied automatically. It is easy to verify that the corresponding multidimensional random process $\|\xi_{ij}(t)\|$ $(i = 1, 2, \dots, k;\ j = 1, 2, \dots, m)$, defined in the space Ω, will be a homogeneous Markov process. Denote by $\mathbf{P}\{\|s_{ij}\|;\, t\}$ the probability that at time t the system is in the state $\|s_{ij}\|$. In this case, from the transitivity of the Markov process under consideration (all states of the process communicate), there follows the existence and uniqueness of the limiting probabilities

$$\mathbf{P}\{\|s_{ij}\|\} = \lim_{t\to\infty} \mathbf{P}\{\|s_{ij}\|;\, t\}, \qquad \|s_{ij}\| \in \Omega. \tag{3}$$

To obtain the probabilities $\mathbf{P}(s_{1\cdot}, \dots, s_{k\cdot})$, $\mathbf{P}(s_{\cdot 1}, \dots, s_{\cdot m})$, $\mathbf{P}(s_{1\cdot}, \dots, s_{k\cdot},\ s_{\cdot 1}, \dots, s_{\cdot m})$, it is sufficient to sum the probabilities of all those matrices $\|s_{ij}\|$ for which the sums of the elements are prescribed, respectively, in each row, in each column, and in the last case—both in the rows and in the columns of the matrix $\|s_{ij}\|$.

4. Since in the case $l \geq k$ the switches of the first stage operate independently of one another, the joint distribution of the numbers of busy lines in the first-stage switches is equal to the product of the corresponding probabilities for each of the switches:

$$\mathbf{P}\{(s_{1\cdot}, \dots, s_{k\cdot}); t\} = \prod_{i=1}^{k} \mathbf{P}\{s_{i\cdot}; t\}.$$

Since each of the $\binom{m}{s_{i\cdot}}$ variants of occupancies of prescribed $s_{i\cdot}$ out of m connecting paths has the same probability, we have

$$\mathbf{P}\{\|s_{ij}\|; t\} = \frac{\prod_{i=1}^{k} \mathbf{P}\{s_{i\cdot}; t\}}{\prod_{i=1}^{k} \binom{m}{s_{i\cdot}}}.$$

Passing to the limit as $t \to \infty$, we obtain

$$\mathbf{P}\{\|s_{ij}\|\} = \frac{\prod_{i=1}^{k} \mathbf{P}(s_{i\cdot})}{\prod_{i=1}^{k} \binom{m}{s_{i\cdot}}}, \qquad \|s_{ij}\| \in \Omega, \tag{4}$$

where $\mathbf{P}(s_{i\cdot})$ is determined by the well-known Erlang formulas ((1), Chap. 6):

$$\mathbf{P}(s_{i\cdot}) = \frac{\dfrac{\lambda^{s_{i\cdot}}}{s_{i\cdot}!}}{\sum_{\nu=0}^{n} \dfrac{\lambda^{\nu}}{\nu!}}, \qquad 0 \leq s_{i\cdot} \leq n, \quad i = 1, 2, \dots, k. \tag{5}$$

5. The principal aim of the present note is the derivation of an approximate formula for $\mathbf{P}(s_{\cdot 1}, \dots, s_{\cdot m})$, valid for large values of k.

For this purpose, consider a sequence of mutually independent, identically distributed m-dimensional random vectors $\vec{\xi}^{(i)} = (\xi_{i1}, \xi_{i2}, \dots, \xi_{im})$, whose components take the values 0 and 1 and satisfy condition (1), where

$$\mathbf{P}\{\xi_{ij} = s_{ij};\ j = 1, 2, \dots, m\} = \frac{\mathbf{P}(s_{i\cdot})}{\binom{m}{s_{i\cdot}}}, \quad i = 1, 2, \dots, k \tag{6}$$

Compute the moments of the first and second orders of the random vector $\vec{\xi}^{(i)}$. By the theorem of total probability we have

$$\mathbf{E}\xi_{i\alpha} = \sum_{\nu=1}^{n} \mathbf{P}(\nu)\frac{\nu}{m} = \frac{\lambda}{m}[1-\mathbf{P}(n)] = a, \quad \alpha = 1, 2, \dots, m. \tag{7}$$

The variances and covariances of the vector $\vec{\xi}^{(i)}$ are, respectively,

$$\mathbf{D}\xi_{i\alpha} = \frac{\lambda}{m}[1-\mathbf{P}(n)]\left[1-\frac{\lambda}{m}+\frac{\lambda}{m}\mathbf{P}(n)\right] = \sigma^2, \quad \alpha = 1, 2, \dots, m; \tag{8}$$

$$\mathbf{E}\{(\xi_{i\alpha}-a)(\xi_{i\beta}-a)\} = \frac{\lambda^2}{m(m-1)}[1-\mathbf{P}(n-1)-\mathbf{P}(n)] - a^2 = r, \tag{9}$$

$$\alpha, \beta = 1, 2, \dots, m, \quad \alpha \neq \beta.$$

Denote the matrix of central second moments of the vector $\vec{\xi}^{(i)}$ by A. By symmetry of the distribution,

$$A = \begin{Vmatrix} \sigma^2 & r & \dots & r \\ r & \sigma^2 & \dots & r \\ \vdots & \vdots & \ddots & \vdots \\ r & r & \dots & \sigma^2 \end{Vmatrix}.$$

Since $\operatorname{Det} A = [\sigma^2 + (m-1)r](\sigma^2 - r)^{m-1} \neq 0$, the matrix A^{-1} exists and is equal to

$$A^{-1} = \|b_{\alpha\beta}\|, \quad \alpha, \beta = 1, 2, \dots, m,$$

$$b_{\alpha\beta} = \frac{1}{\sigma^2 - r}\left[\delta_{\alpha\beta} - \frac{r}{\sigma^2 + (m-1)r}\right], \quad \delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases} \tag{10}$$

Consider the random vector

$$\vec{\eta}^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)}, \dots, \eta_m^{(k)}) = \sum_{i=1}^{k} \vec{\xi}^{(i)}.$$

Denote

$$\mathbf{P}\{\eta_1^{(k)} = s_{\cdot 1}, \dots, \eta_m^{(k)} = s_{\cdot m}\} = \mathbf{P}_k(s_{\cdot 1}, \dots, s_{\cdot m}).$$

Putting

$$u_\alpha = \frac{\eta_\alpha^{(k)} - ka}{\sqrt{k}}, \quad \alpha = 1, 2, \ldots, m,$$

we obtain

$$\mathbf{E}u_\alpha = 0, \quad \mathbf{D}u_\alpha = \sigma^2, \quad \mathbf{E}u_\alpha u_\beta = r, \quad \alpha, \beta = 1, 2, \ldots, m, \quad \alpha \neq \beta.$$

From the multidimensional local limit theorem (see (3); (2), Ch. I) it follows that, as $k \to \infty$,

$$k^{m/2}\mathbf{P}_k(s_{\cdot 1}, \ldots, s_{\cdot m}) \sim \frac{1}{(2\pi)^{m/2}(\mathrm{Det}\, A)^{1/2}} \exp\left\{-\frac{1}{2}\sum_{\alpha,\beta=1}^{m} u_\alpha b_{\alpha\beta} u_\beta\right\}. \tag{11}$$

Thus, for large k, relation (11) can be used to estimate the probability of the joint distribution of the numbers $s_1, \ldots, s_m$.

6. In a number of cases it may be useful to apply the statistic

$$\sum_{\alpha,\beta=1}^{m} u_\alpha b_{\alpha\beta} u_\beta,$$

which is a function of the random variables $\eta_\alpha^{(k)}$ $(\alpha = 1, 2, \ldots, m)$ and has a limiting χ^2 distribution with m degrees of freedom (see ([4]), Ch. 30; ([5])). Using expression (10) for $b_{\alpha\beta}$, we write this statistic in the form

$$\sum_{\alpha,\beta=1}^{m} u_\alpha b_{\alpha\beta} u_\beta = \frac{1}{\sigma^2 - r}\left[\sum_{\alpha=1}^{m} u_\alpha^2 - \frac{r}{\sigma^2 + (m-1)r}\left(\sum_{\alpha=1}^{m} u_\alpha\right)^2\right]. \tag{12}$$

In statistical practice it is customary to use the χ^2 criterion if all expected frequencies are not less than 10 ([4]). In the case under consideration this requirement reduces to the condition $E\eta_\alpha^{(k)} = \frac{k\lambda}{m}[1 - P(n)] \geq 10$ $(\alpha = 1, 2, \ldots, m)$, or, neglecting the probability of losses $P(n)$, to the condition $k \geq 10m/\lambda$.

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Note: Figure translations are in progress. See original paper for figures.