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Abstract

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ON A SPECIAL CASE OF THE RIEMANN–HILBERT PROBLEM

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1. The Riemann–Hilbert problem (problem A) in an $(m+1)$-connected domain G, bounded by the contour $\Gamma = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_m$ (Γ_j are smooth, nonintersecting curves), consists in finding functions $\varphi(z)$, holomorphic in G and continuous in $G + \Gamma$, satisfying the boundary condition

$$\mathrm{Re}(\bar{\lambda}\varphi) = \mu(t), \tag{1}$$

where λ and μ are prescribed functions of the point $t \in \Gamma$, $|\lambda| \equiv 1$. Below we study the homogeneous problem A_0 ($\mu \equiv 0$) in the case when the index of the problem

$$n = \frac{1}{2\pi} \Delta_\Gamma \arg \lambda$$

satisfies the inequality $0 \le n \le m-1$. We shall call this case of the problem the **special case** of the Riemann–Hilbert problem.

For a multiply connected domain, problem A was studied in detail in ([1,2]). In ([2]) the homogeneous adjoint problem—the problem A_0^*—was introduced and the relation

$$l - l^* = 2n - m + 1 \tag{2}$$

was established between l and l^*, the numbers of linearly independent solutions of the problems A_0 and A_0^*, respectively. For $n < 0$ or $n > m-1$, from (2) follows the basic formula

$$l = \max(0, 2n - m + 1). \tag{3}$$

The special case of the problem A_0 differs qualitatively from the other cases, when $n < 0$ or $n > m-1$. In particular, formula (3) may then turn out to be incorrect for some individual problems A_0. Until very recently the special case had not been studied. Only recently have a number of results in this direction been obtained by I. N. Vekua ([3]). Below we present some other results on this question. In particular, we establish that formula (3) remains valid for the "overwhelming majority of problems A_0."

2. In view of the fact that we are interested only in the qualitative picture of the solvability of problem A_0, it is convenient and natural to reduce the problem A_0 to canonical form. Therefore we shall assume that G is a circular domain, with Γ_0 the unit circle, and $\Gamma_j,\ j \geq 1$, circles inside Γ_0, and that the point $z = 0$ belongs to G. By $P_n(z_0)$ we shall denote the class of functions analytic in G, continuous on Γ, and having a pole of order not exceeding n at the fixed point $z_0 \in G$. For $z_0 = 0$ we put $P_n(0) = P_n$. By means of a number of simple substitutions, problem A_0 can be brought to the following form:

Problem B_0. Determine $\omega(z) \in P_n$ from the boundary condition

$$\operatorname{Re}\,[e^{ic(t)}\omega(t)] = 0 \quad \text{on } \Gamma, \tag{4}$$

where $c(t) = c_j = \text{const}$ on $\Gamma_j,\ j \geq 1,\ c_0 = 0$.

In what follows, the sequence $c_j,\ j \geq 1$, will be regarded as a point of the m-dimensional torus T_m and denoted by c. The passage from problem A_0 to the corresponding problem B_0, i.e., from the coefficient λ to the point $c \in T_m$, will briefly

denoted by U_G, $c = U_G(\lambda)$. Thus, the operation U_G maps the set of all problems A_0 onto the set of all problems B_0, i.e. onto the torus T_m. Every problem A_0 with index n has an equivalent representative among the problems B_0. In what follows we shall study the solvability relations of the problem B_0.

The problem conjugate to the problem B_0 will be the problem B_0'.

Problem B_0'. Find $v(z) \in P_{n^*}$, $n^* = m - n - 1$, from the boundary condition

$$\operatorname{Re}\,[e^{-ic(t)}t^{m-1}t'v] = 0 \quad \text{on } \Gamma. \tag{5}$$

The numbers l_n and l_n' (the numbers of linearly independent solutions of the problems B_0 and B_0') for any n and any c are connected by relation (2). By R_n we shall denote the set of those points $c \in T_m$ for which formula (3) holds. Let $CR_n = T_m - R_n$. In the nonspecial case CR_n is the empty set. In the special case, as will be clear from what follows, CR_n is not empty for any n, $0 \leq n \leq m-1$, nor for any domain G. It is not difficult to verify that a meromorphic function in G satisfying boundary condition (4) can be analytically continued across Γ. By $N_G(a)$ we shall denote the number of a-points of the analytic function $\varphi(z)$ located inside the domain G, and by $N_\Gamma^j(a)$ the number of a-points of $w(z)$ on Γ_j;

$$N_\Gamma(a) = \sum_{j=0}^{m} N_\Gamma^j(a).$$

Lemma 1. *For any meromorphic function $w(z)$ in G having no poles on Γ and satisfying boundary condition (4), the numbers $N_G(a)$ and $N_\Gamma(a)$ are finite and satisfy the relation*

$$2N_G(a) + N_\Gamma(a) = 2N_G(\infty),$$

where $N_G(\infty)$ is the sum of the multiplicities of all poles of $w(z)$ inside G; $N_\Gamma^j(a)$ are even for all j and a.

Lemma 1 is proved with the aid of the argument principle. It follows from Lemma 1 that any solution of the problem B_0 in the class P_0 is a constant. Such a solution is nontrivial only under the condition $c_j = 0$, $j \geq 1$ (1, 3).

Theorem 1. *For $n = 1$, every solution of the problem B_0 for which $w(0) = \infty$ realizes a conformal one-to-one mapping of the domain G onto a region of the w-plane with cuts along the straight lines $\operatorname{Re}(e^{ic_j}w) = 0$, $0 \leq j \leq m$. Any two such solutions w and v are connected by the relation $w = Cv + iC_0$, where C and C_0 are real constants; $C_0 \neq 0$ only under the condition $c_j = 0$, $j = 1, \dots, m$.*

We note that, by virtue of Lemma 1, one of the indicated cuts may pass through the origin. The first part of the theorem follows directly from Lemma 1. To prove the second, consider the complex function $f(v) = w(v^{-1})$. It maps the v-plane with cuts along the straight lines $\operatorname{Re}(e^{ic_j}v) = 0$ one-to-one onto the w-plane with cuts along the same straight lines, and $f(\infty) = \infty$. Next consider the function

$$f_1(v) = \frac{f(v) - f(0)}{v}.$$

On the basis of Lemma 1 it is proved that $f_1(v)$ is bounded everywhere. $f_1(v)$ is a solution of the inhomogeneous Dirichlet problem

$$\operatorname{Im} f_1 = -\operatorname{Im} \frac{f(0)}{v}.$$

It is shown that this problem is solvable only under the condition

$$\operatorname{Im} \frac{f(0)}{v} = 0.$$

This, obviously, completes the proof. For any n and c the following inequalities are easily verified:

$$l_n \leq l_{n+1} \leq l_n + 2, \qquad l'_{n+1} \leq l'_n \leq l'_{n+1} + 2, \tag{6}$$

$$\max(0, 2n - m + 1) \leq l_n \leq m \quad \text{for } 0 \leq n \leq m - 1. \tag{7}$$

(7) follows from (2), (6), and the indicated consequence of Lemma 1. From (2) and (6) the theorem is obtained directly:

Theorem 2. *If for some c and $n < m - 1$*

$$l_n = 2n - m + 1,$$

then for all $n' > n$

$$l_{n'} = 2n' - m + 1.$$

Theorem 3. In the special case, for the number of solutions of problem $B_0(A_0)$ the estimate

$$l_n \leqslant n + 1. \tag{8}$$

holds. This estimate is sharp.

Estimate (8) is equivalent to the inequality $l_n + l'_n \leqslant m + 1$.

Let $w_j,\ j = 1, \dots, l_n$, and $v_k,\ k = 1, \dots, l'_n$, be complete systems of solutions of the problems B_0 and B'_0, respectively. Consider the products $\varphi_{jk} = w_j \cdot v_k$. They are solutions of the problem

$$\mathrm{Re}\{iz^{m-1}z'\varphi_{jk}\} = 0$$

in the class P_{m-1}. Let τ be the number of linearly independent solutions of this problem. Since $\Delta_\Gamma \arg z^{m-1}z' = 0$, using (7) it is not hard to verify that $\tau \leqslant m$. But one can show that among the products φ_{jk} there are at least $l_n + l'_n - 1$ linearly independent ones. Hence we obtain $l_n + l'_n \leqslant m + 1$. The sharpness of estimate (8) is shown in item 6 (see also ([3])).

3. For the subsequent derivations it is essential to write the solvability conditions for the problem B_0 in a more suitable form. Let $\Phi_k,\ k = 1, 2, \dots, p$, be a complete system of solutions of the problem

$$\mathrm{Re}\left[e^{-ic(t)}t'\Phi_k\right] = 0$$

in the class P_0. Obviously, $p = m$ or $p = m - 1$. If $c_j = 0,\ j \geqslant 1$, then the functions Φ_k are constructed simply with the aid of the harmonic measures of the curves Γ_k with respect to G ([3]). In the general case they are, obviously, solutions of certain integral equations whose coefficients contain the parameters e^{ic_j}. If the principal part of the Laurent expansion of the sought function in a neighborhood of the point $z = z_0$ is written in the form

$$\sum_{s=1}^{n} \frac{\gamma}{(z - z_0)^s},$$

where γ_s are complex numbers, then by simple calculations one obtains:

Lemma 2. For solvability of the problem B_0 in the class $P_n(z_0)$, it is necessary and sufficient that there exist nontrivial solutions γ of the system of linear equations of the form

$$\mathrm{Re}\left[\sum_{s=1}^{n} \Phi_k^{(s-1)}(z_0)\gamma_s\right] = 0 \tag{9}$$

provided that not all $c_j = 0$.

The matrix of the system (9) in the real domain is equivalent to the matrix

$$A_n = \{\Delta_n, \overline{\Delta}_n\},$$

where Δ_n is a rectangular matrix,

$$\Delta_{ks} = \Phi_k^{(s-1)}(z_0), \quad s = 1, \dots, n, \quad k = 1, \dots, p,$$

and $\overline{\Delta}_n$ is the conjugate matrix.

From the assumptions we have made concerning the domain G it follows:

Lemma 3. $\Delta_n = \Delta_n(z_0, c)$ is a holomorphic function of the point $z_0 \in G$ and an analytic, in the sense of a real variable, function of the point $c \in T_m$.

If $c \neq 0$ and r is the rank of the matrix A_n, then the number of linearly independent solutions of the problem is equal to $2n - r$. Let $n < m/2$. Denote by $\varphi_n(z_0, c)$ the sum of the squares of the moduli of all minors of order $2n$ of the matrix A_n. By Lemma 3, $\varphi_n(z_0, c)$ is an analytic function of the point $c \in T_m$. It turns out that $\varphi_n(z_0, c)$ does not vanish identically for any $z_0 \in G$. To be convinced of this, it is evidently sufficient to indicate, for the given domain G, at least one problem A_0, $n < m/2$, for which $l_n = 0$. But such a problem necessarily exists among the problems

$$\mathrm{Re}\,[(t-b)^{-n}\varphi] = 0, \quad \varphi \in P_0, \quad n < m/2, \quad b \in G.$$

This fact was proved by I. N. Vekua ([3]). It follows from the linear independence of the functions Φ_k (for $c = 0$). Obviously, the set CR_n coincides with the set of zeros of the function $\varphi_n(z_0, c)$ plus the point $c = 0$. If $n > m/2$, then R_n is obtained from R_{n^*}, $n^* = m - n - 1 < m/2$, by a transformation of the form $c' = -c + c_0$, $c \in R_n$, $c' \in R_{n^*}$, where $c_0 \in T_m$. This follows from simple properties of the operation U_G. Thus, the following theorem has been proved:

Theorem 4. CR_n is the zero level set of a certain analytic (in the real domain) function of the point $c \in T_m$, not identically zero on T_m. If $c \in R_n$, then the number of linearly independent solutions of the problem B_0 in the class P_n is calculated by formula (3).

In particular, R_n, $0 \leqslant n \leqslant m-1$, is open and everywhere dense in T_m. The set CR_n, generally speaking, consists of piecewise-smooth manifolds (possibly intersecting and degenerating to a point) of dimension less than m. Since, for the overwhelming majority of problems A_0, for a given n, $U_G(\lambda) = c \in R_n$, Theorem 4 establishes that for "typical" problems A_0, i.e., always, apart from rare exceptions, when $U_G(\lambda) \in R_n$, the number of linearly independent solutions of the problems A_0 should be counted by formula (3). From the openness of the sets R_n it follows that, if for a given problem A_0 formula (3) is valid, then for all problems $\widetilde{A}_0$ sufficiently "close" to the problem A_0, formula (3) remains valid.

The necessary and sufficient condition for solvability of the problem B_0 can also be formulated in the following way. By F_n^c, $c = (c_0, c_1, \dots, c_m)$, denote the class of Riemann surfaces n-sheetedly covering some domain of the complex plane, bounded by cuts of finite length, situated along the straight lines $\mathrm{Re}(e^{ic_j}w) = 0,\ 0 \leqslant j \leqslant m$. The edge of the surface F_n^c is projected onto these straight lines. Then from Lemma 1 one obtains:

Corollary. *The problem B_0 admits a solution having at the point $z = 0$ a pole of exactly order n if and only if the domain G admits a conformal mapping onto some surface of class F_n^c.*

In this connection we note that from Theorem 1 it follows that CR_0 consists of the single point $c = (0, \dots, 0)$, while CR_1 is a certain continuous image of the domain G with the point $z = 0$ removed; the dimension of $CR_1 \leqslant 2$. From the corollary given above it is geometrically clear that CR_n is the image of the n-fold taken domain $G - (0)$; its dimension $\leqslant 2n$.

4. Let n be a given number $\geqslant m/2$. From Lemma 1 and the right-hand side of (7) one can obtain the following corollary for the theory of conformal mappings: having prescribed in advance $c = (c_0, c_1, \dots, c_m)$, any $(m+1)$-connected domain G can be mapped conformally onto some surface of class $F_{n'}^c$, $n' \leqslant n$, and in such a way that a preassigned point $z_0 \in G$ goes to the point $w = \infty$. If $n > m - 1$, then $n' = n$, and the number of such linearly independent mappings is equal to $2n - m + 1$. The last assertion makes it possible to fix the projections of the images of a certain number of interior and boundary points of the domain G ([3]).

5. The results given above also carry over to the theory of the generalized Schottky problem, in which it is required to find meromorphic functions in G admitting poles of prescribed order at a finite number of interior, preassigned points of the domain G, and satisfying the boundary condition (4) ([4]). It is obvious that the qualitative conclusions from Theorem 4 also carry over to the problem A_0, for $0 \leqslant n \leqslant m-1$, in the class of generalized analytic functions.

6. If one first maps the domain G conformally onto a canonical domain with radial cuts so that the point $z = 0$ goes to the point $w = \infty$, then in some cases the problem B_0 will admit polynomial solutions. These polynomial solutions are easily found. If, in particular, the domain G is a domain with cuts along the real axis, then the solutions of the problem B_0 for $c_j = 0,\ j = 1, \dots, m$, will be the polynomials $i,\ iz,\ iz^2, \dots,\ iz^n$. There are exactly $n+1$ linearly independent of them. By virtue of Theorem 3 these solutions form a complete system of solutions of our problem.

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Note: Figure translations are in progress. See original paper for figures.