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Abstract

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MATHEMATICS

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SOLUTION OF A MIXED BOUNDARY-VALUE PROBLEM IN THE THEORY OF THERMODIFFUSION

(Presented by Academician A. A. Dorodnitsyn, December 3, 1957)

In the present article a method is proposed for solving a mixed boundary-value problem for a system of two equations of Fourier type, occurring in the theory of thermodiffusion (1). The solution is represented in the form of a sum of integrals containing arbitrary functions of the integration parameter. The latter are determined from the boundary conditions by solving a system of Volterra integral equations.

1°. Consider the system of equations

$$\frac{\partial u_1}{\partial t} = a_1 \frac{\partial^2 u_1}{\partial x^2} + b_1 \frac{\partial u_2}{\partial t}, \quad \frac{\partial u_2}{\partial t} = a_2 \frac{\partial^2 u_2}{\partial x^2} + b_2 \frac{\partial^2 u_1}{\partial x^2}, \quad (1)$$

$$(a_1 + b_1 b_2 + a_2)^2 \neq 4a_1 a_2$$

with initial conditions

$$u_i(x, 0) = f_i(x), \quad 0 < x < l, \quad (2)$$

and boundary conditions

$$\begin{aligned} a_{11}u_1(0, t) + a_{12}u_2(0, t) + a_{13} \frac{\partial u_1(0, t)}{\partial x} + a_{14} \frac{\partial u_2(0, t)}{\partial x} &= \chi_1(t), \\ a_{21}u_1(0, t) + a_{22}u_2(0, t) + a_{23} \frac{\partial u_1(0, t)}{\partial x} + a_{24} \frac{\partial u_2(0, t)}{\partial x} &= \chi_2(t), \\ a_{31}u_1(l, t) + a_{32}u_2(l, t) + a_{33} \frac{\partial u_1(l, t)}{\partial x} + a_{34} \frac{\partial u_2(l, t)}{\partial x} &= \chi_3(t), \\ a_{41}u_1(l, t) + a_{42}u_2(l, t) + a_{43} \frac{\partial u_1(l, t)}{\partial x} + a_{44} \frac{\partial u_2(l, t)}{\partial x} &= \chi_4(t), \end{aligned} \quad (3)$$

where a_i, b_i, a_{kl} are constant coefficients; $f_i(x), \chi_k(t)$ ($i = 1, 2; k, l = 1, 2, 3, 4$) are arbitrary bounded integrable functions of their arguments. We shall regard the unknown functions

$$u_i(0, t) = \varphi_i(t), \quad u_i(l, t) = \psi_i(t) \quad (4)$$

as given; their determination will be considered later.

By the combined application of the Fourier and Laplace transforms, the solution of the problem (1), (2), (4) can be obtained in the form

$$u_i(x, t) = \sum_{s,j}^{1,2} (A_{sj}^i V_{sj} + B_{sj}^i W_{sj}), \quad (5)$$

where A_{sj}^i, B_{sj}^i are constant coefficients;

$$V_{sj} = -\frac{1}{l} \int_0^t \frac{\partial}{\partial x} \vartheta_3 \left[\frac{x}{2l}, \frac{\mu_s(t-\tau)}{l^2} \right] \varphi_j(\tau) d\tau + \frac{1}{l} \int_0^t \frac{\partial}{\partial x} \vartheta_3 \left[\frac{l-x}{2l}, \frac{\mu_s(t-\tau)}{l^2} \right] \psi_j(\tau) d\tau,$$

$$W_{sj} = \frac{1}{2l} \int_0^l \left\{ \vartheta_3 \left[\frac{x-\xi}{2l}, \frac{\mu_s t}{l^2} \right] - \vartheta_3 \left[\frac{x+\xi}{2l}, \frac{\mu_s t}{l^2} \right] \right\} f_j(\xi) d\xi,$$

$$\mu_s = \frac{1}{2} \left[a_1 + b_1 b_2 + a_2 + (-1)^{s+1} \sqrt{(a_1 + b_1 b_2 + a_2)^2 - 4a_1 a_2} \right],$$

$$\vartheta_3(x, t) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} \exp\{-(x+n)^2/t\}.$$

2°. Let us proceed to determine the unknown functions $\varphi_i(t), \psi_i(t)$ from the boundary conditions (3). Differentiating (5) with respect to x and replacing the second derivative with respect to x of the theta function ϑ_3 by its derivative with respect to t , we obtain

$$\frac{\partial u_i(x, t)}{\partial x} = \sum_{s,j=1}^2 (A_{sj}^i P_{sj} + B_{sj}^i Q_{sj}), \quad (6)$$

where

$$P_{sj} = -\frac{1}{\mu_s l} \int_0^t \frac{\partial}{\partial(t-\tau)} \vartheta_3 \left[\frac{x}{2l}, \frac{\mu_s(t-\tau)}{l^2} \right] \varphi_j(\tau) d\tau +$$

$$+ \frac{1}{\mu_s l} \int_0^t \frac{\partial}{\partial(t-\tau)} \vartheta_3 \left[\frac{l-x}{2l}, \frac{\mu_s(t-\tau)}{l^2} \right] \psi_j(\tau) d\tau, \quad (7)$$

$$Q_{sj} = \frac{1}{2l} \int_0^l \frac{\partial}{\partial x} \left\{ \vartheta_3 \left[\frac{x-\xi}{2l}, \frac{\mu_s t}{l^2} \right] - \vartheta_3 \left[\frac{x+\xi}{2l}, \frac{\mu_s t}{l^2} \right] \right\} f_j(\xi) d\xi.$$

Taking into account the above representation of the function ϑ_3 , it is not difficult to see that the derivative $\frac{\partial}{\partial(t-\tau)} \vartheta_3$, which exists for all $x \neq 0$ in the first and for all $x \neq l$ in the second of the integrals on the right-hand side of equality (7), tends to infinity as $t^{-3/2}$ when $t \rightarrow 0$, if $t \rightarrow \tau$ with $x = 0$ in the first and with $x = l$ in the second of the mentioned integrals.

To eliminate this difficulty, let us replace conditions (3) by others, which are obtained from (3) by integration with respect to t . Integrating (6) with respect to t , we find

$$\int_0^t \frac{\partial u_i(x, \tau)}{\partial x} d\tau = \sum_{s,j=1}^2 (A_{sj}^i P_{sj}^* + B_{sj}^i Q_{sj}^*), \quad (8)$$

where

$$P_{sj}^* = \int_0^t P_{sj}(x, \tau) d\tau, \quad Q_{sj}^* = \int_0^t Q_{sj}(x, \tau) d\tau.$$

Integrating (7) with respect to t and changing the order of integration by means of Dirichlet's formula, we obtain

$$\begin{aligned} P_{sj}^*(x, T) &= -\frac{1}{\mu_s l} \int_0^T \varphi_j(\tau) d\tau \int_\tau^T \frac{\partial}{\partial(t-\tau)} \vartheta_3 \left[\frac{x}{2l}, \frac{\mu_s(t-\tau)}{l^2} \right] dt + \\ &+ \frac{1}{\mu_s l} \int_0^T \psi_j(\tau) d\tau \int_\tau^T \frac{\partial}{\partial(t-\tau)} \vartheta_3 \left[\frac{l-x}{2l}, \frac{\mu_s(t-\tau)}{l^2} \right] dt. \end{aligned}$$

Hence it follows that for all $0 < x < l$ we have

$$\begin{aligned} P_{sj}^*(x, T) &= -\frac{1}{\mu_s l} \int_0^T \varphi_j(\tau) \vartheta_3 \left[\frac{x}{2l}, \frac{\mu_s(T-\tau)}{l^2} \right] d\tau + \\ &+ \frac{1}{\mu_s l} \int_0^T \psi_j(\tau) \vartheta_3 \left[\frac{l-x}{2l}, \frac{\mu_s(T-\tau)}{l^2} \right] d\tau. \end{aligned} \quad (9)$$

By direct calculations, as was done in (2), it can be shown that formula (9) is valid for $x = 0$, $x = l$. Finding expressions (8) for these values of x and substituting them into the equations obtained after integrating (3) with respect to t leads to a system of Volterra integral equations of the first kind

$$\sum_{\beta=1}^4 \int_0^t K_{\alpha\beta}(t, \tau) \varphi_{\beta}(\tau) d\tau = g_{\alpha}(t) \quad (\alpha = 1, 2, 3, 4), \quad (10)$$

where $\varphi_3 = \psi_1$, $\varphi_4 = \psi_2$. The functions $K_{\alpha\beta}(t, \tau)$ and $g_{\alpha}(t)$ have a form that makes it possible to reduce system (10) to a system of generalized Abel integral equations (3), which can easily be reduced to a system of Volterra integral equations of the second kind

$$\varphi_{\alpha}(t) + \sum_{\beta=1}^4 \int_0^t \theta_{\alpha\beta}(t, \tau) \varphi_{\beta}(\tau) d\tau = \omega_{\alpha}(t) \quad (\alpha = 1, 2, 3, 4), \quad (11)$$

where $\theta_{\alpha\beta}(t, \tau)$ and $\omega_{\alpha}(t)$ are continuous functions of the variables t and τ . The solution of the posed problem is obtained after determining, by a known method (4), the functions $\varphi_{\alpha}(t)$ from the system of equations (11) and substituting them into equalities (5).

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Note: Figure translations are in progress. See original paper for figures.

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