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## Abstract

## Full Text

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## A COMPLEX OF LINES OF THE SIXTH DEGREE GENERATED BY A TETRAHEDRAL COMPLEX

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In this work we consider the construction of a complex of lines of the 6th degree $\Sigma^6$, generated by a tetrahedral quadratic complex $\Sigma^2$, and obtain some properties of this complex. As is known ([1]), the complex $\Sigma^6$ has not been considered in the literature, and therefore both its synthetic construction and the properties following from this construction are of interest.

For a tetrahedral quadratic complex $\Sigma^2$ the following two theorems hold:

**Theorem 1.** *The cones of the complex $\Sigma^2$ with vertices on a line passing through one of the principal points of the complex intersect the principal plane of the complex opposite to it in one and the same curve of the 2nd order.*

**Theorem 2.** *The cones of the complex $\Sigma^2$ with vertices at points lying on a plane passing through one of the principal points of the complex intersect the principal plane of the complex opposite to it in curves of the 2nd order belonging to one pencil.*

One of the base points of this pencil is not a principal point of the complex $\Sigma^2$. We shall regard this point as corresponding to the given plane.

Consider an arbitrary line $g$ of space. Let $X$ be an arbitrary point of this line. The rays of the complex $\Sigma^2$ passing through the point $X$ form a cone of the 2nd order $X^2$. Through the line $g$, two tangent planes $\alpha$ and $\beta$ can be drawn to the cone $X^2$.

$$X^2 \cap \alpha \equiv l; \quad X^2 \cap \beta \equiv m;$$

$l$ and $m$ are rays of the complex $\Sigma^2$. The rays of the complex $\Sigma^2$ lying in the plane $\alpha$ envelop a curve of the 2nd order $\alpha^2$; the rays of the complex $\Sigma^2$ lying in the plane $\beta$ envelop a curve of the 2nd order $\beta^2$. The ray $l$ touches $\alpha^2$ at the point $X \subset g$; the ray $m$ touches $\beta^2$ likewise at the point $X$. Let $X_1$ be the second point of intersection of the curve $\alpha^2$ with the line $g$; $X_2$ the second point of intersection of the curve $\beta^2$ with the line $g$. Then to the arbitrary point $X$ of the line $g$ there correspond two points $X_1$ and $X_2$. This correspondence is a $[2, 2]$-valued involutory correspondence $X \leftrightarrow (X_1, X_2)$.

We set ourselves the following problem:

**Problem.** To study the set $\Sigma$ of lines $g$ of space such that on each of them the correspondence $X \leftrightarrow (X_1, X_2)$ reduces to the correspondence $X \leftrightarrow (X_1 \equiv X_2)$.

Let the line $g$ belong to the set $\Sigma$; then to each point $X \subset g$ there corresponds a unique point $X_1 \equiv X_2$, belonging to $g$, such that the cones $X^2$ and $X_1^2$ of the complex $\Sigma^2$ have two common tangent planes.

Further, relying on Theorems 1 and 2, one may draw the following conclusion: all lines passing through the point of intersection of $g$ with the principal plane $\gamma$ of the complex $\Sigma^2$ and lying in the plane formed by $g$ with the principal point $A$ of the complex $\Sigma^2$ opposite to $\gamma$, will belong to $\Sigma$. Let the cones of the complex $\Sigma^2$ with vertices on the line $g$ intersect the principal pla -

the plane $\gamma$ along curves of the 2nd order belonging to the pencil $|K^2|$. Then on the plane there exist $\infty^1$ pairs of lines (each such pair is obtained from the intersection with the plane $\gamma$ of two common tangent planes to each pair of cones $X^2$ and $X_1^2$) issuing from the point $O = g \cap \gamma$, such that each pair is tangent to two curves of the pencil $|K^2|$. The converse will also be true, i.e., if from the point $O$ one can draw $\infty^1$ pairs of lines such that each pair is tangent to two curves of the pencil $|K^2|$, then there exists a pencil of lines $g$ with center at the point $O$, belonging to the set $\Sigma$, and the plane of this pencil will pass through the point $A$.

The following problem is posed:

**Problem 1.** Given an arbitrary point $O$ on the plane $\gamma$. Determine the number of planes from the pencil with carrier $OA$ such that the lines passing through the point $O$ and lying in them belong to $\Sigma$.

To the pencil of planes with carrier $AO$ there corresponds, on the plane $\gamma$, a curve of the 2nd order $K^2$, passing through the point $O$ and the principal points of the complex $\Sigma^2$ lying on the plane $\gamma$.

Now Problem 1 can be reduced to the following:

**Problem 2.** On the curve of the 2nd order $K^2$, determine the number of points for which the corresponding planes have the property that the lines lying in them and passing through the point $O$ belong to the set $\Sigma$.

On the basis of what was said above, Problem 2 can be reduced to the following:

**Problem 3.** On the curve of the 2nd order $K^2$, determine the number of such points $Y$ that each of them, together with the principal points of the complex $\Sigma^2$ lying on the plane $\gamma$, determines a pencil of curves of the 2nd order $|Y^2|$, possessing the property that there exist $\infty^1$ pairs of lines issuing from the points $O$, such that each pair is tangent to two curves of the pencil $|Y^2|$.

Before proceeding to the solution of the question posed, let us dwell on the solution of the following problem:

**Problem.** A pencil of curves of the 2nd order is given by its base points. Find the locus of points $X$ possessing the following property: from the point $X$ one can draw $\infty^1$ pairs of lines, each of which is tangent to two curves of the given pencil.

A known theorem states: if a secant intersects a conic section at two points $A$ and $A_1$, and two tangents to this same conic section at the points $B$ and $B_1$ and the chord of contact of these tangents at the point $C$, then $C$ is the double point of the involution determined by the pairs of corresponding points $A, A_1$ and $B, B_1$. Relying on this theorem, we find that the required locus of points $X$ is the diagonal lines of the complete quadrilateral formed by the base points of the given pencil.

Using the result obtained in solving Problem 3, we find that on the curve of the 2nd order $K^2$ there exist three points satisfying the condition of this problem.

Now it may be said that there exist three planes from the pencil with carrier $OA$, where $O$ is an arbitrary point of the plane $\gamma$, satisfying the condition of Problem 1. The latter says that the set $\Sigma$ is a complex.

On an arbitrary line $m$ of the plane $\gamma$ there exist three points $D, D_1, D_2$, possessing the property that the lines passing through them and lying in the planes $|ma|$ belong to the complex $\Sigma$.

When the line $m$ rotates about an arbitrary point $O \in \gamma$, the points $D, D_1, D_2$ describe curves of the 2nd order passing through the point $O$ and the principal points of the complex $\Sigma^2$ lying on the plane $\gamma$.

From what was said above it follows that the rays of the complex $\Sigma$ passing through an arbitrary point of space form three cones of the 2nd order passing through the principal points of the complex $\Sigma^2$. Thus the complex $\Sigma$ is a complex of the 6th order.

By Sturm's definition, the degree of a complex is equal to its order; consequently, the complex $\Sigma$ is a complex of the 6th degree $\Sigma^6$.

On the basis of what has been set forth, we obtain the following properties of the complex $\Sigma^6$:

1. The complex $\Sigma^6$ contains the rays of the 4 bundles and 4 fields formed by the vertices and planes of the principal tetrahedron of the complex $\Sigma^2$.
2. Let $A$ and $\gamma$ be a principal point and the principal plane opposite to it of the complex $\Sigma^2$. In every plane of the bundle $\{A\}$ there lie three pencils of rays of the complex $\Sigma^6$, whose vertices lie on the line of intersection of the plane under consideration with the plane $\gamma$.
3. Through an arbitrary point of the plane $\gamma$ there pass three planes of the bundle $\{A\}$, in which lie pencils of rays of the complex $\Sigma^6$ with centers at the given point. The rays of the complex $\Sigma^6$ passing through an arbitrary point of space form a cone of the 6th order, decomposing into three cones of the 2nd order passing through the vertices of the principal tetrahedron of the complex $\Sigma^2$.

4. The cones of the complex $\Sigma^6$ with vertices on a line passing through the point $A$ intersect the plane $\gamma$ in three curves of the 2nd order belonging to one pencil.

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## CITED LITERATURE

1. *Encyclopädie der mathematischen Wissenschaften*, 3, H. 7, Leipzig, 1920, S. 1084–1166.

*Note: Figure translations are in progress. See original paper for figures.*