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Abstract

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ON CERTAIN CLASSES OF MODELS

In the note ([1]) a structural characterization of quasivarieties of classes of algebras was indicated. Using this result, we indicate below structural characterizations of universally axiomatizable classes of models and of quasiprimitive classes of algebras. Thereby the question, left open in ([2]), of a purely algebraic internal characterization of quasiprimitive classes of algebraic systems is also solved simultaneously. At the end it is shown that, up to structural equivalence, the only axiomatizable homomorphically closed classes of models admitting a theory of defining relations in the sense of ([3]) are the quasiprimitive classes of algebraic systems.

All the categories of structures considered below will be assumed to have strong substructures, and direct compositions, when they exist, will be assumed to coincide with direct products ([3]).

1. We shall call a category of structures K a **category with finitary homomorphisms** if, whatever local system of K-substructures $\mathfrak{A}_\alpha$ covering a K-structure $\mathfrak{B}$ may be, every mapping of a structure $\mathfrak{A}$ into a K-structure $\mathfrak{B}$ which is, for each substructure $\mathfrak{A}_\alpha$, a homomorphism of $\mathfrak{A}_\alpha$ into a suitable K-substructure of the structure $\mathfrak{B}$, is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. In accordance with the usual terminology of group theory, we shall call a K-structure $\mathfrak{A}$ **locally finite** if every finite set of elements of $\mathfrak{A}$ lies in a suitable finite K-substructure. It is clear that all categories of models have finitary homomorphisms. The following theorem is also easily proved.

Theorem 1. *Every category of structures K with finitary homomorphisms and locally finite structures is structurally equivalent to a suitable class of models.*

We shall call a category of structures K **locally consistent** if, from the fact that every finite subsystem of an arbitrary system of K-structures $\mathfrak{S}$, defined on subsets of some set, is embeddable in a K-structure as its substructures, it follows that the whole system $\mathfrak{S}$ is embeddable in a suitable K-structure. From this definition, in particular, it follows that in locally consistent categories every increasing chain of K-structures embedded one in another can be embedded in a containing structure. The local theorem of the restricted predicate calculus shows that all axiomatizable categories of models are locally consistent.

Recall that a class of models K is called **universally axiomatizable** if it can be characterized by some set of universal axioms of the form
$(x_1)\dots(x_n)\mathfrak{P}(x_1,\dots,x_n)$, where the expression $\mathfrak{P}$ contains no quantifiers.

Theorem 2. *In order that a category K be structurally equivalent to some universally axiomatizable class of models, it is necessary and sufficient that K be locally consistent, have finitary homomorphisms, and that every subset of a K-structure be its K-substructure.*

Theorem 3. *If a universally axiomatizable class of models with a finite set of basic predicates $P_1, \dots, P_k$ is structurally equivalent to a class of models with basic predicates $Q_1, \dots, Q_l$, then there are equivalences of the form*

$$P_i(x_1, \dots, x_{m_i}) \sim \mathfrak{P}_i(x_1, \dots, x_{m_i}),$$

$$Q_j(x_1, \dots, x_{n_j}) \sim \mathfrak{Q}_j(x_1, \dots, x_{n_j}),$$

where $\mathfrak{P}_i, \mathfrak{Q}_j$ are open formulas with predicate variables Q_j, respectively P_i, and with the equality predicate.

In the case where the number of basic predicates is infinite, Theorem 3 is also true, but in the role of $\mathfrak{P}_i, \mathfrak{Q}_j$ one must allow also infinite expressions.

2. A model $\mathfrak{A}$ with basic predicates $P_1, P_2, \dots$ of arities $n_1, n_2, \dots$ is called an **algebraic system** of type $\tau = \langle I;\ n_1, n_2, \dots \rangle$, where I is some part of the set of indices of the predicates, if the predicates P_i for $i \in I$ are predicates of operations on $\mathfrak{A}$. The class A_τ of all algebraic systems of type τ is a class that is restrictive, multiplicatively and homomorphically closed in itself, regular, and contains the one-element model. In (1) the notions of a quasifree and a free subclass of a category of structures K were introduced. If K is a category of models, then quasifree and free subclasses singled out from K by some system of axioms, i.e. axiomatizable within K, will be called respectively **quasiprimitive** and **primitive** in K. Quasiprimitive (primitive) subclasses of the class K will simply be called **quasiprimitive (primitive) classes of algebraic systems** of the given type. From the theorems of Tarski–Łoś (4,5) and Birkhoff (6) it follows directly that a subclass L of some class of models K is quasiprimitive in K if and only if L can be singled out from K by axioms of the form

$$(x_1) \dots (x_n)(R_1 \& \dots \& R_s \supset R_{s+1}),$$

where the R_i are expressions of the form

$$P_\alpha(x_{i_1}, \dots, x_{i_n}).$$

Let the category K: α) be multiplicatively closed; β) contain the one-element structure. Then the intersection of any system of quasifree (free) subclasses in K will be a quasifree (free) subclass. Therefore, for each class T of K-structures in K there is a least quasifree (free) subclass L containing T. We shall call the class L the **quasifree (free) closure** of T in K and write $L = T^q$ ($L = T^f$). It is easy to see that T^q consists of all possible K-substructures of direct products of T-structures. To obtain an analogous characterization of T^f, impose on K the further requirements: γ) K is homomorphically closed in itself, and δ) the preimage of a K-substructure of an arbitrary K-structure $\mathfrak{A}$ under a homomorphism onto $\mathfrak{A}$ of any K-structure $\mathfrak{B}$ is a K-substructure in $\mathfrak{B}$. Then the free closure L^f of a quasifree subclass L will consist of all possible K-structures that are homomorphic images of L-structures. Hence it follows that if K and T are axiomatizable classes of models, then T^q and T^f will also be axiomatizable. Further, if the category of structures K satisfies conditions $\alpha) - \delta)$ and is regular, then every L^f-free structure will belong to L, where L is a quasifree subclass in K. In particular, if L^f contains free structures with any number of L^f-free generators, then the stock of free structures does not change in passing from L to its free closure.

Theorem 4. *Let a regular category K, satisfying $\alpha) - \delta)$, contain a finite structure $\mathfrak{A}$. Then: 1) K-free structures with different finite numbers of free generators are nonisomorphic; 2) in the minimal quasifree $\{\mathfrak{A}\}^q$ and free $\{\mathfrak{A}\}^f$ subclasses containing $\mathfrak{A}$, every structure with a finite set of generators is finite; 3) if the number of nonisomorphic K-structures of each finite cardinality is finite, then $\{\mathfrak{A}\}^f$ contains only a finite number of non-one-element minimal quasifree and minimal free subclasses.*

Assertions 1), 3) are generalizations of the theorems of Fudzhivar ([7]) and Scott ([8]), proved by them for primitive classes of algebras.

3. Let Γ be a partially ordered set, any two elements of which have a common larger element. Suppose that with each $\alpha \in \Gamma$ there is associated an object $\mathfrak{A}_\alpha$ of a category K, and with each pair $\langle \alpha, \beta \rangle$ $(\alpha, \beta \in \Gamma,\ \alpha < \beta)$ there is associated a homomorphism $\pi_{\alpha\beta} : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$, in such a way that from $\alpha < \gamma < \beta$ it follows that $\pi_{\alpha\beta} = \pi_{\alpha\gamma}\pi_{\gamma\beta}$. It is said that Γ and the mappings $\alpha \to \mathfrak{A}_\alpha$, $\langle \alpha, \beta \rangle \to \pi_{\alpha\beta}$ constitute a **direct spectrum**. An object $\mathfrak{A}$ of the category K, with given homomorphisms $\pi_\alpha : \mathfrak{A}_\alpha \to \mathfrak{A}$, is called the **limit of the spectrum** ([9]) if $\pi_\alpha = \pi_{\alpha\beta}\pi_\beta$ $(\alpha < \beta)$, and if for any system of homomorphisms σ_α of the objects $\mathfrak{A}_\alpha$ into an arbitrary K-object $\mathfrak{B}$, satisfying the conditions $\sigma_\alpha = \pi_{\alpha\beta}\sigma_\beta$ $(\alpha < \beta)$, there is one and only one homomorphism $\xi : \mathfrak{A} \to \mathfrak{B}$ for which $\sigma_\alpha = \pi_\alpha\xi$ $(\alpha \in \Gamma)$. In the case where K is a category of structures, it will henceforth be assumed, without further stipulation, that Γ has a least element 0 and that the mappings $\pi_{\alpha\beta}$ are homomorphisms of $\mathfrak{A}_\alpha$ onto $\mathfrak{A}_\beta$. Then it may be considered that the structures $\mathfrak{A}_\alpha$ and $\mathfrak{A} = \lim \mathfrak{A}_\alpha$ are given on $\mathfrak{A}_0$ with a suitably defined equality relation (cf. ([11])). If K is the category of all models of a fixed type, then for any direct spectrum, under the indicated restrictions, the limiting model exists and its construction is described in ([11]). It is shown there as well that if a universal or positive axiom holds on all models of the spectrum, then it holds also on the limiting model. Since all universally axiomatizable classes of algebraic systems are characterized by positive and universal axioms, every such class contains the limits of spectra of its systems ([11]). The converse of this is given by the following theorem.

Theorem 5. *A multiplicatively closed class of algebraic systems that contains* *all subsystems of its systems is axiomatizable if and only if it contains the limits of direct spectra of its systems.*

In particular, a quasifree class of algebraic systems is quasiprimitive if and only if it contains the limits of direct spectra of its systems. Taking into account Theorem 5 from ([1]), we obtain: in order that a category of structures K be structurally equivalent to an axiomatizable class of algebras that is multiplicatively closed and contains subalgebras of its algebras, it is necessary and sufficient that K be a category with divisible homomorphisms, multiplicatively and homomorphically closed in itself, restricted, regular, additive, and that in K there exist limits of direct spectra of the type indicated above. Adding to these conditions the requirement that a one-element structure exist, we obtain a structural characterization of quasiprimitive classes of algebras.

Let K be an arbitrary category of structures and let $\mathfrak{A} \in K$. An equivalence relation θ, defined on the underlying set of $\mathfrak{A}$, will be called a **congruence** on $\mathfrak{A}$ (cf. ([10])) if θ belongs to some homomorphism of $\mathfrak{A}$ onto a suitable K-structure. We shall call the equivalence θ an **external congruence** if, for any two homomorphisms σ, ρ of an arbitrary K-structure $\mathfrak{B}$ into $\mathfrak{A}$, from the relations $b_\alpha^\sigma \equiv b_\alpha^\rho(\theta)$ for some generating system of elements b_α of the structure $\mathfrak{B}$, it follows that $b^\sigma \equiv b^\rho(\theta)$ for all $b \in \mathfrak{B}$.

It is obvious that, *in order that a quasifree class of algebras K be free, it is necessary and sufficient that in K every external congruence be a congruence.*

Thus, in order to obtain an internal structural characterization of primitive classes of algebras, it is enough to adjoin to the set of structural properties indicated above, which characterize quasifree classes of algebras, the requirement that external congruences coincide with congruences.

4. We shall call a class of models K a **class with local embeddability** if, from the fact that every finite submodel of an arbitrary model $\mathfrak{M}$ is isomorphically embeddable in a suitable K-model, it follows that the model $\mathfrak{M}$ itself is embeddable in a suitable K-model.

Lemma 1. *Let a class of models K with local embeddability contain K-free models with any finite number of free, densely generating elements. Then, for every operation $\Phi_{\alpha n}$ (see (1)), on all K-models the formula*

$$\Phi_{\alpha n}(x_1, \dots, x_n) = x_{n+1} \sim (\exists x_{n+2}) \cdots (\exists x_s)\, \mathfrak{A}_{\alpha n}(x_1, \dots, x_s), \tag{1}$$

is valid, where $\mathfrak{A}_{\alpha n}$ is a suitable conjunction of terms of the form $P_\alpha(x_{i_1}, \dots, x_{i_k})$.

Lemma 2. *If a class of models K with local embeddability in itself is homomorphically closed and contains K-free models with any cardinal number of K-finitarily dense free elements, then, by adding Φ-operations to the number of the basic predicates of the class K, we turn it into a universally axiomatizable class of algebraic systems.*

On the basis of these lemmas the following can now be proved:

Theorem 6. *Every R-complete, homomorphically closed, axiomatizable class of models K is structurally equivalent to a quasiprimitive class of algebraic systems.*

Indeed, from the R-completeness of K it follows[3] that in K there exist K-free models with any cardinal number of free generators. From the homomorphic closedness of K it follows (((1), Theorem 1) that nonempty intersections of K-submodels of K-models are K-submodels. By virtue of the main result of note [12], in view of the axiomatizability of K, this entails the additivity of K. Theorem 2 of [1] shows that the free generators of K-free models will be finitarily dense. Finally, by virtue of Lemma 2, we conclude that the enrichment of K by the Φ-operations defined by formulas (1) gives a quasiprimitive class of algebraic systems.

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Note: Figure translations are in progress. See original paper for figures.