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Abstract

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ON THE REDUCIBILITY OF A CANONICAL SYSTEM OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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In various questions of mechanics, in particular in the theory of gyroscopes, an important role is played by the investigation of solutions of the canonical system of differential equations

$$\frac{dq_r}{dt} = \frac{\partial \mathscr{H}}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial \mathscr{H}}{\partial q_r} \qquad (r = 1, 2, \dots, m), \tag{1}$$

where q_r and p_r are, respectively, generalized coordinates and generalized impulses, and the Hamiltonian $\mathscr{H}$ is a quadratic form in them with periodic coefficients of the form

$$\mathscr{H} = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(\omega t) q_i q_j + \sum_{i,j=1}^{m} b_{ij}(\omega t) q_i p_j + \frac{1}{2}\sum_{i,j=1}^{m} c_{ij}(\omega t) p_i p_j. \tag{2}$$

(ω is a large parameter.)

In the present work the following problem is posed: to find a transformation of the variables q_i and p_i that reduces the initial differential system with variable coefficients to a new, likewise canonical, system of differential equations with constant coefficients.

To solve the indicated problem, we first write the system of equations (1) in the form

$$\frac{dx}{dt} = JH(\omega t)x. \tag{3}$$

Here $x = (q_1, q_2, \dots, q_m, p_1, p_2, \dots, p_m)$ is a vector of order $2m$; $H(\omega t)$ is a real symmetric periodic matrix-function; the matrix J has the form

$$J = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix},$$

where E_m denotes the identity matrix of order m.

If in equation (3) we make the change of independent variable, putting $\tau = \omega t$, then we arrive at the vector differential equation

$$\frac{dx}{d\tau} = \varepsilon J H(\tau) x, \qquad \varepsilon = \omega^{-1}. \tag{4}$$

In the last equation we make a transformation of the unknown function, setting $x = U(\tau, \varepsilon) y$, where $U(\tau, \varepsilon)$ is a matrix-function periodic with respect to

τ of period 2π, and we shall require that y satisfy the differential equation

$$\frac{dy}{d\tau} = \varepsilon J \widetilde{H} y, \tag{5}$$

in which $\widetilde{H}$ is a matrix with constant elements. Let us note that if the matrix $\widetilde{H}$ turns out to be symmetric, then it will thereby be proved that, by means of the indicated transformation, the Hamilton equations (1) with variable coefficients are reduced to Hamilton equations with constant coefficients.

In order that the substitution $x = U(\tau, \varepsilon) y$ reduce equation (4) to the form (5), the transformation function $U(\tau, \varepsilon)$ must satisfy the equation

$$\frac{dU}{d\tau} = \varepsilon \big(J H(\tau) U - U J \widetilde{H} \big). \tag{6}$$

In the paper [1] the existence was proved of a matrix-function $U(\tau, \varepsilon)$, satisfying the condition $U(0, \varepsilon) = E$, periodic in τ, analytic with respect to ε (uniformly with respect to τ), and such that, by means of the change of variable $x = U(\tau, \varepsilon) y$, the original equation (4) is reduced to an equation of the form (5). There it was also proved that the matrix $\widetilde{H}$ is analytic with respect to the parameter ε, and also that the solutions of the original equation (4) are analytic with respect to the parameter ε.

Thus, here it remains for us only to show that, by means of the indicated transformation of equation (4), we arrive at equation (5), in which the matrix $\widetilde{H}$ will be symmetric.

Proceeding to the proof of the symmetry of the matrix $\widetilde{H}$, let us note that for $\tau = 0$ we have the obvious equality

$$U^* J U - J = 0, \tag{7}$$

where U^* is the matrix transposed with respect to the matrix U. We shall show that the derivative of the left-hand side of the last equality with respect to τ, for any τ, is equal to zero; consequently, it is identically zero, and at the same time, for $\tau = 0$, we shall prove the symmetry of the matrix $\widetilde{H}$.

Indeed,

$$\frac{d}{d\tau}(U^*JU - J) = \left(\frac{d}{d\tau}U^*\right) JU + U^*J\left(\frac{d}{d\tau}U\right).$$

Substituting here the expressions for the derivatives of the matrices U and U^* from the identities

$$\frac{dU}{d\tau} = \varepsilon(JH(\tau)U - UJ\widetilde{H}), \qquad \frac{dU^*}{d\tau} = \varepsilon(-U^*H(\tau)J + \widetilde{H}^*JU^*),$$

we obtain

$$\frac{d}{d\tau}(U^*JU - J) = \varepsilon(\widetilde{H}^*JU^*JU - U^*JUJ\widetilde{H}).$$

After simple transformations the last expression can be written in the form

$$\frac{d}{d\tau}(U^*JU - J) = \varepsilon[(\widetilde{H} - \widetilde{H}^*) + \widetilde{H}^*J(U^*JU - J) - (U^*JU - J)J\widetilde{H}]. \tag{8}$$

By virtue of the periodicity of the function U, from the last equality we have

$$\widetilde{H} - \widetilde{H}^* = \frac{1}{2\pi}\int_0^{2\pi} [(U^*JU - J)J\widetilde{H} - \widetilde{H}^*J(U^*JU - J)]\, d\tau. \tag{9}$$

Further differentiating equality (8) with respect to the variable ε, we obtain

$$\begin{aligned}\frac{d}{d\tau}\left\{\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right\} + \varepsilon\left\{\left[\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right]J\widetilde{H} - \widetilde{H}^*J\left[\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right]\right\}\\ = \varepsilon\left\{\frac{\partial}{\partial\varepsilon}(\widetilde{H} - \widetilde{H}^*) - (U^*JU - J)J\frac{\partial\widetilde{H}}{\partial\varepsilon} + \frac{\partial\widetilde{H}^*}{\partial\varepsilon}J(U^*JU - J)\right\}\\ + \left\{(\widetilde{H} - \widetilde{H}^*) - (U^*JU - J)J\widetilde{H} + \widetilde{H}^*J(U^*JU - J)\right\}.\end{aligned} \tag{10}$$

Before proceeding to estimate the solutions of the differential equation (10), let us prove the following lemma:

Lemma. Suppose a differential equation is given

$$\frac{dz}{dt} = L(z) + f(t), \qquad z(0) = 0, \tag{11}$$

in which the function $L(z)$ on every finite interval $a \leqslant t \leqslant b$ satisfies the condition $|L(z)| \leqslant \lambda |z|$, $\lambda = \text{const}$. Then there exists a constant M such that $|z| \leqslant M\|f\|$, where $\|f\| = \max_{a\leqslant t\leqslant b} |f(t)|$.

Proof. We replace the differential equation (11) by the equivalent integral equation:

$$z = \int_0^t L(z)\,dt + \int_0^t f(t)\,dt, \tag{12}$$

whence

$$|z| \leqslant \lambda \int_0^t |z|\,dt + t\|f\|. \tag{13}$$

Put

$$y = \lambda \int_0^t |z|\,dt, \qquad \frac{dy}{dt} = \lambda |z|.$$

Multiplying inequality (13) by λ, we obtain

$$\frac{dy}{dt} - \lambda y \leqslant \lambda \|f\| t.$$

Hence

$$\frac{d}{dt}(ye^{-\lambda t}) \leqslant \lambda \|f\| t e^{-\lambda t}.$$

Integrating from 0 to t, we obtain

$$ye^{-\lambda t} \leqslant \|f\| \left[-te^{-\lambda t} + \frac{1 - e^{-\lambda t}}{\lambda} \right]; \qquad y \leqslant \|f\| \left[-t + \frac{e^{\lambda t} - 1}{\lambda} \right].$$

Noting that $|z| \leqslant y + \|f\| t$, we finally have

$$|z| \leqslant \|f\| \left\{ \frac{e^{\lambda t} - 1}{\lambda} \right\} \quad \text{or} \quad |z| \leqslant M\|f\|, \quad \text{where} \quad M = \max_{a\leqslant t\leqslant b} \left\{ \frac{e^{\lambda t} - 1}{\lambda} \right\}.$$

To estimate the solution of equation (10), let us note that for $\varepsilon = 0$ it follows from equality (6) that $U = E$; consequently, for $\varepsilon = 0$ we have $U^*JU - J = 0$; on the other hand, for $\varepsilon = 0$, $U^*JU - J = 0$. Therefore the proved lemma on estimates of solutions of a differential system is applicable to equation (10), and hence

$$\left\|\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right\| \leqslant \varepsilon M_1 \left\|\frac{\partial}{\partial\varepsilon}(\widetilde{H} - \widetilde{H}^*)\right\| + M_2\|U^*JU - J\| + M_3\|\widetilde{H} - \widetilde{H}^*\|,$$

where M_1, M_2, M_3 are certain constants.

Substituting into the last inequality the expression for $\widetilde{H} - \widetilde{H}^*$ from (9), and also for $\frac{\partial}{\partial\varepsilon}(\widetilde{H} - \widetilde{H}^*)$, after differentiating equality (9) with respect to the parameter ε, we obtain

$$\left\|\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right\| \leqslant \varepsilon N_1 \left\|\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right\| + N_2\|U^*JU - J\|, \tag{14}$$

where N_1 and N_2 are new constants. (14) can be rewritten in the form

$$\left\|\frac{\partial}{\partial\varepsilon}(U^*JU - J)\right\| \leqslant N_3\|U^*JU - J\|, \qquad \text{where} \qquad N_3 = \frac{N_2}{1 - \varepsilon N_1}, \quad \varepsilon N_1 < 1.$$

But, since for $\varepsilon = 0$ we have $U^*JU - J = 0$, it follows from the last inequality that $U^*JU - J \equiv 0$. Then it follows from inequality (9) that $\widetilde{H} \equiv \widetilde{H}^*$, i.e., the matrix $\widetilde{H}$ is symmetric. Thus the symmetry of the matrix $\widetilde{H}$, and consequently the reducibility of the differential equations of canonical form, has been proved.

In view of the analyticity, proved above, of the matrices $U_n(\tau, \varepsilon)$ and $\widetilde{H}(\varepsilon)$ with respect to the parameter ε, they are represented by expansions of the form

$$U(\tau, \varepsilon) = E + \sum_{n=1}^{\infty} \varepsilon^n U_n(\tau), \qquad \widetilde{H}(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \widetilde{H}_n,$$

convergent for sufficiently small values of the parameter ε.

To determine the coefficients $U_n(\tau), \widetilde{H}_n$ of these expansions, we substitute the indicated expansions into the differential equation (6) and compare, in the resulting identity, the coefficients of like powers of ε. Using, further, at each stage the periodicity of $U_n(\tau)$, we shall have

$$\widetilde{H}_0 = \frac{1}{2\pi}\int_0^{2\pi} H(\tau)\,d\tau; \qquad U_1 = J\int_0^{\tau} (H(\tau) - \widetilde{H}_0)\,d\tau;$$

$$\widetilde{H}_1 = \frac{1}{2\pi}\int_0^{2\pi}(H(\tau)U_1 + JU_1J\widetilde{H}_0)\,d\tau; \qquad U_2 = J\int_0^{\tau}(H(\tau)U_1 + JU_1J\widetilde{H}_0 - \widetilde{H}_1)\,d\tau;$$

$$\widetilde{H}_2 = \frac{1}{2\pi}\int_0^{2\pi}(H(\tau)U_2 + JU_2J\widetilde{H}_0 +$$

$$+JU_1J\widetilde{H}_1)\,d\tau; \qquad U_3 = J\int_0^{\tau}(H(\tau)U_2 + JU_2J\widetilde{H}_0 +$$

$$+JU_1JH_1 - \widetilde{H}_2)\,d\tau;$$

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Thus, the solution of the equation

$$\frac{dx}{d\tau} = \varepsilon JH(\tau)x$$

has the form

$$x = \left[E + \sum_{n=1}^{\infty}\varepsilon^n U_n(\tau)\right]y,$$

where y is a solution of the equation $\frac{dy}{d\tau} = \varepsilon J\widetilde{H}(\varepsilon)y$ with constant coefficients, in which $\widetilde{H}(\varepsilon)$, in turn, is represented by an expansion convergent for sufficiently small ε, i.e., for sufficiently large ω:

$$\widetilde{H}(\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n\widetilde{H}_n,$$

and $\widetilde{H}_s = \widetilde{H}_s^*$.

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1. K. A. Breus, DAN, **108**, No. 6 (1956); Ukr. Math. Zh., **10**, issue 2 (1958).

Note: Figure translations are in progress. See original paper for figures.

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