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Abstract

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ON THE DEPENDENCE BETWEEN THE NUMBER OF SIGN CHANGES OF A SOLUTION OF AN ELLIPTIC EQUATION AND THE GROWTH OF THE SOLUTION

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Let there be given a linear elliptic equation of the second order

$$Lu \equiv \sum_{i,k=1}^{n} a_{ik}(x_1, \dots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x_1, \dots, x_n) \frac{\partial u}{\partial x_i} + c(x_1, \dots, x_n)u = 0, \tag{1}$$

defined in the spherical layer $D = \{r_1 < |x| < r_2\}$, where $x = (x_1, \dots, x_n)$. Let $c(x) \le 0$. Suppose, further, that the coefficients a_{ik}, $i, k = 1, \dots, n$, of equation (1) are twice continuously differentiable in D, and the coefficients b_i, $i = 1, \dots, n$, are continuously differentiable in D. Suppose that in D the inequalities

$$|a_{ik}|,\ |\partial a_{ik}/\partial x_j|,\ |\partial^2 a_{ik}/\partial x_j \partial x_l|,\ |b_i|,\ |\partial b_i/\partial x_j|,\ |c| < M; \quad i, j, k, l = 1, \dots, n \tag{2}$$

hold.

Suppose that in D the condition of uniform ellipticity is satisfied,

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \Big/ \sum_{i=1}^{n} \xi_i^2 > a > 0. \tag{3}$$

Let $u(x)$ be a solution of equation (1) in D, continuous in $\overline{D}$. Denote by D^+ and D^-, respectively, the sets of points $x \in D$ at which $u(x) > 0$ and $u(x) < 0$. We shall call a component of the set D^+ or D^- **essential** if it has limit points both on the inner and on the outer spheres bounding the spherical layer D. The total number of all essential components of the sets D^+ and D^- will be called the **number of sign changes** of the function $u(x)$ in the spherical layer D.

Theorem 1. *There exists a constant C, depending on the constant M in inequality (2) and on the constant a in inequality (3), such that for every solution $u(x)$ of equation (1) in the spherical layer $D = \{r_1 < |x| < r_2\}$, where $r_2 < 1$, continuous in $\overline{D}$ and having N sign changes, at least one of the following two inequalities holds:*

$$\max_{|x|=r_2} |u(x)| \Big/ \max_{|x|=\sqrt{r_1 r_2}} |u(x)| > \left(\frac{r_2}{r_1}\right)^{N^{\frac{1}{n-1}}/C}$$

or

$$\max_{|x|=r_1} |u(x)| \Big/ \max_{|x|=\sqrt{r_1 r_2}} |u(x)| > \left(\frac{r_2}{r_1}\right)^{N^{\frac{1}{n-1}}/C}.$$

The proof of this theorem is based on the following lemmas.

Lemma 1. *Let $K \subset D$ be a ball with center at the point O. Let $G \subset K$ be a domain containing the point O and having limit points on the boundary of the ball K. There exists a constant C_1, depending on the constan-*

of M in inequality (2) and on the constant a of inequality (3), such that, whenever for the domain G the inequality

$$\mu_n G < \mu_n K / C_1 \tag{4}$$

is satisfied (by $\mu_k E$ we shall denote the k-dimensional measure of the set E), and $u(x)$ is a solution of equation (1), defined in G and vanishing on that part of the boundary of G which lies strictly inside K, then

$$\sup_{x \in G} u(x) \geq 2u(0).$$

Proof. Without loss of generality, one may assume that $u(x) > 0$ for $x \in G$. Let R be the radius of the ball K. For every r, $0 < r < R$, denote by K_r the ball of radius r with center at the point O, and by S_r the surface of the ball K_r. Put $G_r = G \cap K_r$ and $\Gamma_r = G \cap S_r$. From inequality (4) it follows that there is an r_0, $0 < r_0 < R$, such that

$$\mu_{n-1} \Gamma_{r_0} < \mu_{n-1} S_{r_0} / C_1. \tag{5}$$

By Serrin' s theorem [(1)] there exists a function $K(x, x')$, $x \in K_{r_0}$, $x' \in S_{r_0}$, such that for any continuous function $\varphi(x')$, defined on S_{r_0}, the function $v(x)$, defined by the equality

$$v(x) = \int_{S_{r_0}} \frac{K(x,x')\varphi(x')}{\mu_{n-1} S_{r_0}}\, ds, \tag{6}$$

has the properties: 1) $Lv \leq 0$ and 2) $v/S_{r_0} = \varphi$. Moreover

$$0 < K(0,x') < C^*, \tag{7}$$

where C^* is a constant depending on the constants M and a of inequalities (2) and (3). Put $C_1 = 2C^*$. Further put: $\varphi(x') = u(x')$ for $x' \in \Gamma_{r_0}$; $\varphi(x') = 0$ for $x' \in S_{r_0} \setminus \Gamma_{r_0}$. Since on the boundary of G_{r_0} we have the inequality $u(x) \leq v(x)$, by virtue of property 1) of the function v we obtain $u(x) \leq v(x)$ in G_{r_0}. From (5), (6), and (7) we find $v(0) < \sup_{x' \in S_{r_0}} \varphi(x')\, C^*/C_1$, and hence $v(0) < \sup_{x \in S_{r_0}} \varphi(x')/2$, whence $u(0) < \sup_{x \in G} u(x)/2$, as was required to prove.

Lemma 2. Let P be the cylinder $\sum_{k=2}^{n} x_k^2 < h^2 < 1$. Let Γ_1 and Γ_2 be $(n-1)$-dimensional manifolds lying inside the cylinder P, with boundary on the boundary of P, each of which separates, in the cylinder P, points with sufficiently large in absolute value negative coordinates x_1. Let G be the part of P lying between Γ_1 and Γ_2. Let in G equation (1) be defined, satisfying in G conditions (2) and (3). Let $u(x)$ be a solution of the equation, defined in G and continuously differentiable in $\overline{G}$. Suppose that the function $u(x)$ satisfies the conditions: 1) $u|_{\Gamma_1} = u_0 > 0$; 2) $u|_{\Gamma_2} = 0$; 3) $\partial u/\partial n|_{\Gamma_2} \leq 0$, where $\partial/\partial n$ is differentiation with respect to the inner normal, and 4) $u(x) \geq -u_0$ for $x \in G$. Denote by G_1 the aggregate of points $x \in G$ for which $u_0/2 < u(x) < u_0$.

Then $\mu_n G_1 > h^n/C_2$, where C_2 is a constant depending on the constant M of inequalities (2) and on the constant a of inequality (3).

Here we shall indicate the way to prove this lemma. Put $h^n/\mu_{nG} 1 = C_2^*$. Denote by E_t the aggregate of points $x \in G$ for which $u(x) = t$, and by E_t^* the intersection of E_t with the cylinder $\sum_{k=2}^{n} x_k^2 < h^2/4$. There exists a u^*, $u_0/2 < u^* < u_0$, such that: a) the level set $E_{u^*}^*$ contains no points x, where

$\operatorname{grad} u(x) = 0$, and, consequently, consists of smooth $(n-1)$-dimensional manifolds; b) the inequality

$$\int_{E_{u^*}^*} \left|\frac{\partial u}{\partial n}\right| d\sigma > \frac{C_2^* u_0 h^{n-2}}{L_1}, \tag{8}$$

holds, where L_1 is an absolute constant.

Denote by G_{u^*} the set of points $x \in G$ for which $u(x) < u^*$. There is an h^*, $h/2 < h^* < h$, such that, if by S_{h^*} we denote the set of points $x \in G_{u^*}$ for which

$$\sum_{k=2}^{n} x_k^2 = h^{*2},$$

then the inequality

$$\int_{S_{h^*}} \left| \frac{\partial u}{\partial n} \right| d\sigma < L_2 u_0 h^{n-2}, \tag{9}$$

holds, where L_2 is an absolute constant. By condition 3), imposed on the solution $u(x)$, we have

$$\int_{\Gamma_2} \frac{\partial u}{\partial n}\, d\sigma < 0. \tag{10}$$

Consider the equality $\int_{G_{h^*}} Lu\, d\omega = 0$ and apply Green' s formula to its left-hand side. From inequalities (8), (9), (10), (2), (3) and condition 4), taking into account that

$$\left.\frac{\partial u}{\partial n}\right|_{E_{u^*}} < 0$$

(the normal here is internal with respect to G_{u^*}), we obtain

$$(aC_2^*/L_1 - Mn^2 L_2) h^{n-2} u_0 \le L_3 h^{n-1} u_0 + L_4 h^n u_0,$$

where L_3 and L_4 are absolute constants; and since $0 < h < 1$, we arrive at the required estimate of C_2^* in terms of M and a.

With the aid of Lemmas 1 and 2 one proves:

Lemma 3. Let in the spherical layer $D = \{h/4 < |x| < h\}$, $0 < h < 1$, a solution $u(x)$ of equation (1), continuous on $\overline{D}$, be defined. Let $u(x)$ have N changes of sign in D. Let $g_1, g_2, \dots, g_N$ be the essential components of the sets D^+ and D^-. Put

$$m' = \max_{|x|=h/4} |u(x)|, \qquad m'' = \max_{|x|=h/2} |u(x)|, \qquad m''' = \max_{|x|=h} |u(x)|;$$

$$m_i' = \max_{|x|=h/4,\, x\in \overline{g}_i} |u(x)|, \qquad m_i'' = \max_{|x|=h/2,\, x\in \overline{g}_i} |u(x)|, \qquad m_i''' = \max_{|x|=h,\, x\in \overline{g}_i} |u(x)|.$$

There exists a constant C_3, depending on the constant M in inequalities (2) and on the constant a in inequality (3), such that from the fact that

$$m'''/m'' \le 2^{N^{\frac{1}{n-1}}/C_3}, \qquad m'/m'' \le 2^{N^{\frac{1}{n-1}}/C_3} \tag{11}$$

it follows that

$$\min_i m_i'' > 2^{N^{\frac{1}{n-1}}/2^{\frac{3n+1}{n-1}+2} C_1^{\frac{1}{n-1}}} m'', \tag{12}$$

where C_1 is the constant of Lemma 1.

From Lemmas 1 and 3 it follows:

Lemma 4. In the notation of the preceding lemma, the following assertion holds: there exists a constant C_4, depending on the constant M in inequalities (2) and on the constant a in inequality (3), such that

$$m'''/m'' > 2^{N^{\frac{1}{n-1}}/C_4} \quad \text{or} \quad m'/m'' > 2^{N^{\frac{1}{n-1}}/C_4}.$$

Proof. Suppose that inequalities (11) are satisfied and, consequently, by Lemma 3, inequality (12) is satisfied. We have the obvious inequality $\sum_{i=1}^{N} \mu_n g_i < \omega_n h^n$, where ω_n is the volume of the unit n-dimensional ball, and therefore there is an i_0 such that

$$\mu_n g_{i_0} < \omega_n h^n/N. \tag{13}$$

Let

$$N_1 = \left[N^{\frac{1}{n-1}} \Big/ 2^{\frac{3n+1}{n-1}} C_1^{\frac{1}{n-1}} \right].$$

Let N be so large that $N_1 > 3$ (the case when $N_1 \leqslant 3$ is considered separately; in this case the assertion of the lemma is obtained by an argument analogous to the proof of Lemma 1). Put $t_i' = h/2 - h(i-1/2)/4N_1$, $i = 1, \dots, N$; $t_i'' = h/2 + h(i-1/2)/2N_1$, $i = 1, 2, \dots, N$; $m_{k,i_0}' = \max_{|x|=t_k',\, x \in g_{i_0}} |u(x)|$ and $m_{k,i_0}'' = \max_{|x|=t_k'',\, x \in g_{i_0}} |u(x)|$. Suppose these maxima are attained at the points P_{k,i_0}' and P_{k,i_0}''. By the maximum principle, at least one of the following two alternatives holds: $m_{k,i_0}' \leqslant m_{k+1,i_0}'$, $k = 1, 2, \dots, N-1$, or $m_{k,i_0}'' \leqslant m_{k+1,i_0}''$, $k = 1, 2, \dots, N-1$. For definiteness, suppose the first case occurs (the second case is analogous). Denote by K_{k,i_0}' the ball with center at the point P_{k,i_0}' and radius $h/8N_1$, and by g_{k,i_0}' the component of the intersection of g_{i_0} with this ball which contains the point P_{k,i_0}'. Consider those k ($k = 1, 2, \dots, N_1$) for which $\mu_n g_{k,i_0}' < \omega_n h^n / C_1 2^{3n} N_1^n$. The number of such k does not exceed $N_1/2$, since otherwise we would have

$$\mu_n g_{i_0} \geqslant \sum_{k=1}^{N_1} \mu_n g'_{k,i_0} > \frac{N_1}{2}\,\omega_n \frac{h^n}{C_1 2^{3n} N_1^n} \geqslant \omega_n \frac{h^n}{N},$$

which contradicts (13). Consequently, there are at least

$$N_1/2 - 1 \geqslant N^{\frac{1}{n-1}} \Big/ 2^{\frac{3n+1}{n-1}} C_1^{\frac{1}{n-1}}$$

distinct k for which $\mu_n g'_{k,i_0} \geqslant \omega_n h^n / C_1 2^{3n} N_1^n$, i.e. such that Lemma 1 is applicable to g'_{k,i_0}. Hence we obtain

$$m' \geqslant m'_{N_1,i_0} \geqslant 2^{N^{\frac{1}{n-1}} \Big/ 2^{\frac{3n+1}{n-1}+1} C_1^{\frac{1}{n-1}}}\, m'_{1,i_0} >$$

$$> 2^{N^{\frac{1}{n-1}} \Big/ 2^{\frac{3n+1}{n-1}} C_1^{\frac{1}{n-1}}}\, 2^{-N^{\frac{1}{n-1}} \Big/ 2^{\frac{3n+1}{n-1}+2} C_1^{\frac{1}{n-1}}}\, m'' = 2^{N^{\frac{1}{n-1}} \Big/ 2^{\frac{3n+1}{n-1}+2} C_1^{\frac{1}{n-1}}}\, m''.$$

Putting

$$C_4 = \max\left(C_3,\ C_1^{\frac{1}{n-1}} 2^{\frac{3n+1}{n-1}+2}\right),$$

we obtain the constant needed.

Theorem 1 is easily obtained from Lemma 4 if one divides the spherical layer $D = \{r_1 < |x| < r_2\}$ into $[\log_4(r_2/r_1)]$ spherical layers with ratio of radii equal to 4.

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REFERENCES

[1] J. Serrin, *J. Analyse Math.*, 4, No. 2, 292 (1955-1956).

Note: Figure translations are in progress. See original paper for figures.