HYDROMECHANICS

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Abstract

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ON RIEMANN WAVES IN MAGNETIC HYDRODYNAMICS

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Riemann waves in magnetic hydrodynamics were first considered by S. A. Kaplan and K. P. Stanyukovich ([1]) in the case when the magnetic field is parallel to the plane of the wave. Below, Riemann waves in magnetic hydrodynamics are considered for an arbitrary position of the field relative to the plane of the wave front, which leads to the appearance of new mechanical effects.

The equations describing isentropic motions by plane waves of a perfect gas with infinite conductivity, in the presence of a magnetic field, have the form

$$\begin{gathered}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-C\gamma\rho^{\gamma-2}\frac{\partial\rho}{\partial x}+\frac{1}{8\pi\rho}\frac{\partial}{\partial x}\left(H_y^2+H_z^2\right),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}=\frac{H_x}{4\pi\rho}\frac{\partial H_y}{\partial x},\qquad \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}=\frac{H_x}{4\pi\rho}\frac{\partial H_z}{\partial x},\\
\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0,\qquad \frac{\partial H_y}{\partial t}+\frac{\partial}{\partial x}(uH_y-vH_x)=0,\\
\frac{\partial H_z}{\partial t}+\frac{\partial}{\partial x}(uH_z-wH_x)=0,\qquad H_x=\text{const},
\end{gathered}\tag{1}$$

where C is a constant in the adiabatic equation: $p=C\rho^\gamma$.

We shall seek solutions depending on some combination of the independent variables $\varphi(x,t)$. Then the original equations reduce to the form

$$\begin{aligned}
&-u'a+\rho'C\gamma\rho^{\gamma-2}+H_y'\frac{H_y}{4\pi\rho}+H_z'\frac{H_z}{4\pi\rho}=0,\\
&-v'a-H_y'\frac{H_y}{4\pi\rho}=0,\\
&-w'a-H_z'\frac{H_x}{4\pi\rho}=0,\\
&u'\rho-\rho'a=0,\\
&u'H_y-v'H_x-H_y'a=0,\\
&u'H_z-w'H_x-H_z'a=0,
\end{aligned}\tag{2}$$

where the prime denotes differentiation with respect to φ; $a = \lambda - u$; $\lambda = -\dfrac{\partial\varphi}{\partial t}\Big/\dfrac{\partial\varphi}{\partial x}$ is the velocity of motion of the wave.

In order that system (2), linear and homogeneous with respect to the derivatives, have a nontrivial solution, it is necessary that the determinant

of this system, $\Delta(a) = 0$. Solving this equation, we obtain:

$$a_{1,2} = \pm\frac{H_x}{\sqrt{4\pi\rho}},$$

$$a_{3,4,5,6} = \pm\frac{1}{2}\left\{\sqrt{\gamma C\rho^{\gamma-1} + \frac{H^2}{4\pi\rho} + \sqrt{\frac{\gamma C}{\pi}}\,H_x\rho^{1/2\gamma-1}} \pm \sqrt{\gamma C\rho^{\gamma-1} + \frac{H^2}{4\pi\rho} - \sqrt{\frac{\gamma C}{\pi}}\,H_x\rho^{1/2\gamma-1}}\right\}. \tag{3}$$

It follows from this that the a_i are the velocities of propagation of small disturbances ([2]). We shall consider waves traveling through the particles in the positive direction of the x-axis, to which the positive values of a correspond. It is known that the inequalities

$$a_3 \geq a_1 \geq a_4 \geq 0$$

always hold, i.e. waves propagating in one direction cannot overtake one another.

In order to find how the sought quantities change in the wave, it is necessary to solve system (2), substituting into it the corresponding value of a. It is easy to verify that the only solution corresponding to a_1 will be

$$u = u_0, \qquad \rho = \rho_0, \qquad H_y^2 + H_z^2 = H_r^2 = \text{const},$$

$$H_y = H_r\cos\theta, \qquad v = v_0 - \frac{H_r}{\sqrt{4\pi\rho}}\cos\theta, \qquad H_z = H_r\sin\theta, \qquad w = w_0 - \frac{H_r}{\sqrt{4\pi\rho}}\sin\theta,$$

where the quantities with subscript zero are arbitrary constants, and θ is an arbitrary function $\varphi(x,t)$.

Proceeding to the study of other types of waves, let us note that the second, third, fifth, and sixth equations of system (2) may be written in the following form:

$$-aw_1' - (av_1 + \frac{H_x}{4\pi\rho}H_r)\theta' = 0,$$

Fig. 1

Figure 1: Fig. 1

$$-H_{xw}1' - (v_1 H_x + aH_r)\theta' = 0,$$

$$-av_1' - aw_1\theta' - \frac{H_x}{4\pi\rho}H_r' = 0,$$

$$-aH_r' + H_{ru}' - H_{xv}1' - w_1 H_x\theta' = 0,$$

where θ is the angle formed by the vector $\mathbf{H}_r = H_y\mathbf{j} + H_z\mathbf{k}$ with the y-axis; v_1 is the projection of the velocity in the direction of $\mathbf{H}_r$, and w_1 is the projection of the velocity in the direction perpendicular to $\mathbf{H}_r$.

Fig. 1

Let us note that, if $a^2 \neq H_x^2/4\pi\rho$, then the first two equations have the single solution $\theta = \text{const}$, $w_1 = \text{const}$, and the problem reduces to integrating the following system of equations:

$$\begin{aligned} &-u'a && + \rho' C\gamma\rho^{\gamma-2} + H_r'\frac{H_r}{4\pi\rho} = 0, \\ &-v_1'a && - H_r'\frac{H_x}{4\pi\rho} = 0, \\ &u'\rho && - \rho' a = 0, \\ &u'H_r - v_1'H_x && - H_r'a = 0. \end{aligned} \tag{4}$$

Introducing dimensionless variables according to the equalities

$$R = \frac{\rho}{\rho_*}, \qquad U = \frac{\sqrt{4\pi\rho_*}}{H_x}u, \qquad V = \frac{\sqrt{4\pi\rho_*}}{H_x}v_1, \qquad h = \frac{H_r}{H_x},$$

$$A_{3,4} = \frac{\sqrt{4\pi\rho_*}}{H_x}a_{3,4} = \frac{1}{2}\left\{\sqrt{R^{\gamma-1} + \frac{1+h^2}{R} + 2R^{1/2\gamma-1}} \pm \sqrt{R^{\gamma-1} + \frac{1+h^2}{R} - 2R^{1/2\gamma-1}}\right\},$$

where ρ_* is determined from the condition $4\pi\gamma C\rho_*^\gamma = H_x^2$, and combining equations (4), we obtain the following system, containing no parameters:

$$\frac{dh^2}{dR} = 2A^2 - 2R^{\gamma-1}, \qquad \frac{dU}{dR} = \frac{A}{R}, \qquad \frac{dV}{dR} = -\frac{1}{RA}\,\frac{A^2 - R^{\gamma-1}}{h}. \qquad (5)$$

To solve this system it is sufficient to integrate the first equation, after which U and V are found by quadratures. A qualitative picture of the behavior of the integral curves of the first equation (5) for the case $A = A_3$ is shown in Fig. 1, and for the case $A = A_4$ in Fig. 2; moreover, for $A = A_3$ everywhere $dh^2/dR \geq 0$, while for $A = A_4$ everywhere $dh^2/dR \leq 0$. Solutions of equations (5) for weak magnetic fields can be obtained by quadratures.

To obtain a qualitative picture of the deformation of waves with the passage of time, the derivative of the wave propagation velocity $\lambda = u + a$ with respect to density was calculated. It turned out that for $a = a_1$, $d\lambda/d\rho = 0$, while for $a = a_3$ and $a = a_4$, $d\lambda/d\rho \geq 0$, i.e., waves propagating with velocity a_1 are not deformed, while compression waves propagating with velocities a_3 and a_4 have a tendency to turn into a compression shock wave. It follows from this that in self-similar solutions, rotational discontinuities correspond to waves propagating with velocity a_1, while with velocities a_3 and a_4 only rarefaction waves are possible.

Fig. 2

The solutions investigated may be applied to the solution of the problem of the decay of an arbitrary discontinuity in magnetohydrodynamics and of the piston problem; here the piston may move either with constant velocity $ui + vj + wk$, or with a variable velocity chosen in such a way as to excite a wave of only one type, or a sequence of waves such that they do not overtake one another.

In these problems, along with continuous solutions, solutions containing shock waves are possible. It turns out that in the variables in which the Riemann-wave problem was solved, the change of the quantities across a shock wave can be obtained in explicit form by specifying the change in the magnetic-field intensity.

Let us take the conditions on a shock wave in the following form ([2]):

$$j\{v\} = \frac{H_x}{4\pi}\{H_r\}, \qquad j^2\left\{\frac{1}{\rho}H_r\right\} = \frac{H_x^2}{4\pi}\{H_r\}, \qquad j^2 = \frac{p_2 - p_1 + \dfrac{1}{8\pi}\left(H_{r2}^2 - H_{r1}^2\right)}{\dfrac{1}{\rho_2} - \dfrac{1}{\rho_1}},$$

$$\frac{\gamma}{\gamma-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) - \frac{1}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)(p_2 - p_1) + \frac{1}{16\pi}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)(H_{r2} - H_{r1})^2 = 0. \qquad (6)$$

Introducing dimensionless variables

$$h_i = \frac{H_{ri}}{H_x}, \qquad P_i = \frac{4\pi p_i}{H_x^2} = \frac{4\pi C\rho_i^\gamma}{4\pi\gamma C\rho_{*i}^\gamma} = \frac{1}{\gamma}R^\gamma \qquad (i = 1, 2),$$

from the last three equations we obtain

$$\frac{R_1}{R_2} = \frac{h_1\left[(P_2 - P_1) + \frac{1}{2}(h_2^2 - h_1^2)\right] + (h_2 - h_1)}{h_2\left[(P_2 - P_1) + \frac{1}{2}(h_2^2 - h_1^2)\right] + (h_2 - h_1)},$$

$$\frac{\gamma}{\gamma - 1}\left(P_2\frac{R_1}{R_2} - P_1\right) - \frac{1}{2}\left(\frac{R_1}{R_2} + 1\right)(P_2 - P_1) + \frac{1}{4}\left(\frac{R_1}{R_2} - 1\right)(h_2 - h_1)^2 = 0.$$

Substituting the first equality into the second, we obtain a quadratic equation for P_2, solving which we find P_2 as a function of P_1, h_1, h_2. From the first and third equalities (6) we find the change in the quantities U and V.

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CITED LITERATURE

[1] S. A. Kaplan, K. P. Stanyukovich, *DAN*, **96**, No. 3 (1954).
[2] L. D. Landau, E. M. Lifshitz, *Electrodynamics of Continuous Media*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.