Reports of the Academy of Sciences of the USSR

1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1958. Vol. 118, No. 5

MATHEMATICS

B. M. Budak and A. D. Gorbunov

THE METHOD OF LINES FOR SOLVING A NONLINEAR BOUNDARY-VALUE PROBLEM IN A DOMAIN WITH A CURVILINEAR BOUNDARY

(Presented by Academician A. A. Dorodnitsyn on July 17, 1957)

Let it be required to find a continuously differentiable solution of the equation

$$u_{xy} = f(x, y, u, u_x, u_y), \tag{1}$$

the right-hand side of which is defined and continuous in the aggregate of all arguments and satisfies a Lipschitz condition with constant L_1 in u_x and u_y in the closed bounded domain

$$\overline{G}: \quad 0 \le x \le l_x, \quad g(x) \le y \le l_y, \quad |u| \le l_u, \quad |u_x| \le l_{u_x}, \quad |u_y| \le l_{u_y},$$

where $g(x) \ge 0$ for $0 \le x \le l_x$, and $g'(x) \ge 0$ and is continuous; let this solution be required to satisfy the boundary conditions

$$u(x, g(x)) = \varphi(x), \quad 0 \le x \le l_x; \qquad u(0, y) = \psi(y), \quad 0 \le y \le l_y, \tag{2}$$

where $\varphi'(x)$ and $\psi'(y)$ are continuous, and $M_\varphi + 2M_\psi < l_u, \quad M_{\varphi'} + M_{\psi'} M_{g'} < l_{u_x}, \quad M_\psi < l_u, \quad M_{\psi'} < l_{u_y}$*.

The method of lines was previously applied to the solution of an analogous problem for the case in which f does not depend on u_x and u_y.

In §1, under the stated conditions, the existence of at least one solution of problem (1), (2) is established. In §§2 and 3, under additional conditions, the uniqueness of the solution of this problem and its continuous dependence on the boundary conditions are asserted, and an estimate is given for the error incurred when the exact solution is replaced by an approximate one.

§1. Extend f by setting $f(x, y, u, u_x, u_y) = f(x, l_y, u, u_x, u_y)$ for $y \geq l_y$, and $f(x, y, u, u_x, u_y) = f(x, g(x), u, u_x, u_y)$ for $y \leq g(x)$. On each line $x = x_k = kh$ $(k = 0, 1, 2, \dots;\ h > 0)$, replace the sought solution approximately by a function $u_k(y)$, where $u_k(y)$ is determined step by step from the system

$$\frac{\Delta u'_k}{h} = f\left(x_k, y, u_k, \frac{\Delta u_k}{h}, u'_k\right),$$

$$\Delta u_k = u_{k+1}(y) - u_k(y), \qquad \Delta u'_k = u'_{k+1}(y) - u'_k(y), \qquad k = 0, 1, 2, \dots \quad (3)$$

with the boundary conditions

$$u_0(y) = \psi(y), \qquad u'_0(y) = \psi'(y), \qquad u(g(x_k)) = \varphi(x_k), \qquad k = 0, 1, \dots \quad (4)$$

* By M_φ, $M_{\varphi'}$, etc., are denoted the upper bounds of the moduli of φ, φ', etc., in the domain in which they a

Rewriting equation (3) in the form

$$u'_{k+1}(y) = u'_k(y) + hf\left(x_k, y, u_k(y), \frac{u_{k+1}(y) - u_k(y)}{h}, u'_k(y)\right), \qquad (3')$$

we note that the right-hand side of this differential equation with the unknown function $u_{k+1}(y)$ satisfies the Lipschitz condition with constant L_1 with respect to u_{k+1} and is continuous in the aggregate of the arguments y, u_{k+1} for all k and y for which the inequalities

$$0 \leq x_k \leq l_x, \qquad |u_k(y)| \leq l_u, \qquad \left|\frac{u_{k+1}(y) - u_k(y)}{h}\right| \leq l_{u_x}, \qquad |u'_k(y)| \leq l_{u_y}$$

are satisfied; therefore $(3')$ has a unique solution $u'_{k+1}(y)$ with continuous derivative.

Define the domain

$$\bar{G}^*_{xy}: \ 0 \leq x \leq l^*_x, \qquad g(x) - l^*_y \leq y \leq g(x) + l^*_y,$$

in which solutions of the system (3), (4) exist.

Replacing k by i in (3′) and summing over i from 0 to $k-1$, we obtain

$$u_k'(y) = \psi'(y) + h\sum_{i=0}^{k-1} f_i(y), \tag{5}$$

where

$$f_i(y) = f\left(x_i, y, u_i(y), \frac{\Delta u_i(y)}{h}, u_i'(y)\right).$$

Integrating (5) with respect to y from $g(x_k)$ to y, we shall have

$$u_k(y) = \varphi(x_k) + \psi(y) + \psi(g(x_k)) + h\int_{g(x_k)}^{y}\sum_{i=0}^{k-1} f_i(\eta)\,d\eta. \tag{6}$$

Replacing k by $k+1$ in (6), subtracting (6) from the equality obtained and dividing the result by h, we find

$$\frac{\Delta u_k(x)}{h} = \frac{\Delta\varphi(x_k)}{h} - \frac{\Delta\psi(g(x_k))}{h} + \int_{g(x_{k+1})}^{y} f_k(\eta)\,d\eta - \int_{g(x_k)}^{g(x_{k+1})}\sum_{i=0}^{k-1} f_i(\eta)\,d\eta. \tag{7}$$

From (6), (7), and (5) we obtain restrictions on l_x^*, l_y^*:

$$|u_k(\eta)| \le M_\varphi + 2M_\psi + M_f l_x^* l_y^* \le l_u; \tag{8}$$

$$\left|\frac{\Delta u_k(y)}{h}\right| \le M_{\varphi'} + M_{\psi'}M_{g'} + M_f l_y^* + M_f M_g l_x^* \le l_{u_x}; \tag{9}$$

$$|u_k'(y)| \le M_{\psi'} + M_f l_x^* \le l_{u_y}. \tag{10}$$

If l_x^* and l_y^* satisfy conditions (8), (9), (10), and $l_x^* \le l_x$, $l_y^* \le l_y$, then the system (3), (4) will be solvable in the domain $\bar{G}_{xy}^*$.

Next, form the domain

$$\bar{G}_{xy}^{**}:\ 0 \le x \le l_x^{**}, \qquad g(x) \le y \le l_y^*,$$

choosing l_x^{**} so that for $0 \le x \le l_x^{**}$ the inequality $g(x) \le l_y^*$ is satisfied.

Put

$$\tilde{u}(x,y) = u_k(y) + \frac{\Delta u_k(y)}{h}(x - x_k), \qquad x_k \le x \le x_{k+1};$$

$$g(x_{k+1}) - l_y^* \le y \le g(x_k) + l_y^* \quad \text{if } g(x_{k+1}) \ge g(x_k); \tag{11}$$

$$g(x_k) - l_y^* \le y \le g(x_{k+1}) + l_y^* \quad \text{if } g(x_{k+1}) < g(x_k).$$

Then

$$\tilde{u}_x(x,y) = \frac{\Delta u_k(y)}{h}; \tag{12}$$

$$\tilde{u}_y(x,y) = u_k'(y) + \frac{\Delta u_k'(y)}{h}(x - x_k). \tag{13}$$

The functions $\tilde{u}(x,y)$ and $\tilde{u}_y(x,y)$ are continuous everywhere in their domain of definition, while $\tilde{u}_x(x,y)$ is continuous with respect to y and undergoes a finite jump when passing through the lines $x = x_k,\ k = 1, 2, \ldots$.

From (11) it follows that there exists a number h_0 such that every function $\tilde{u}(x,y)$, $0 < h \le h_0$, is defined at least in the domain $\bar{G}_{xy}^{**}$. Differentiating (12) with respect to y, we obtain

$$\tilde{u}_{xy}(x,y) = \frac{\Delta u_k'(y)}{h} = f\left(x_k, y, u_k(y), \frac{\Delta u_k(y)}{h}, u_k'(y)\right), \qquad x_k \le x < x_{k+1}. \tag{14}$$

Therefore one may write

$$\tilde{u}_{xy}(x,y) = f(x, y, \tilde{u}(x,y), \tilde{u}_x(x,y), \tilde{u}_y(x,y)) + \tilde{\Theta}(x,y), \tag{15}$$

where

$$\tilde{\Theta} = f\left(x_k, y, u_k, \frac{\Delta u_k}{h}, u_k'\right) - f(x, y, \tilde{u}, \tilde{u}_x, \tilde{u}_y). \tag{16}$$

By virtue of the continuity of f in the aggregate of its arguments and by virtue of relations (11), (12), (13), we have $\tilde{\Theta} \to 0$ as $h \to 0$. Integrating (15) with respect to x from 0 to x and with respect to y from $g(x)$ to y, we obtain

$$\tilde{u}(x,y) = \tilde{u}(x, g(x)) + \psi(y) - \psi(g(x)) +$$

$$+\int_{g(x)}^{y} d\eta \int_{0}^{x} [f(\xi,\eta,\tilde{u}(\xi,\eta),\tilde{u}_x(\xi,\eta),\tilde{u}_y(\xi,\eta))+\tilde{\Theta}]\, d\xi; \tag{17}$$

$$\tilde{u}_x(x,y)=\tilde{u}_x(x,g(x))+$$

$$+\int_{g(x)}^{y} [f(x,\eta,\tilde{u}(x,\eta),\tilde{u}_x(x,\eta),\tilde{u}_y(x,\eta))+\tilde{\Theta}(x,\eta)]\, d\eta; \tag{18}$$

$$\tilde{u}_y(x,y)=\psi'(y)+\int_{0}^{x} [f(\xi,y,\tilde{u}(\xi,y),\tilde{u}_x(\xi,y),\tilde{u}_y(\xi,y))+\tilde{\Theta}(\xi,y)]\, d\xi. \tag{19}$$

Theorem 1. The families of functions $\tilde{u}(x,y)$, $\tilde{u}_x(x,y)$, $\tilde{u}_y(x,y)$, depending on the parameter h, $0<h\le h_0$, are uniformly bounded and equicontinuous[2] in the domain $\bar{G}^{**}_{xy}$ as $h\to 0$.

Theorem 2. Under the conditions formulated earlier, there exists a sequence $h_\nu \to 0$ as $\nu\to 0$ such that $\tilde{u}^{(\nu)}(x,y)$, $\tilde{u}^{(\nu)}_x(x,y)$, $\tilde{u}^{(\nu)}_y(x,y)$ converge uniformly, respectively, to the continuous functions $u(x,y)$, $v(x,y)$, $w(x,y)$, with $u_x(x,y)=v(x,y)$, $u_y(x,y)=w(x,y)$; $u(x,y)$ satisfies equation (1) and the boundary conditions (2) in the domain $\bar{G}^{**}_{xy}$.

2°. Analogously to how this is done in [2] or [3], the following theorems can be proved:

Theorem 3. If the right-hand side of equation (1) is continuous in the aggregate of all its arguments and satisfies a Lipschitz condition in the last three arguments, then the boundary-value problem (1), (2) in the domain $\bar{G}^{**}_{xy}$ has a unique continuously differentiable solution $u(x,y)$, and $\tilde{u}(x,y)$, $\tilde{u}_x(x,y)$, $\tilde{u}_y(x,y)$ converge uniformly, respectively, to $u(x,y)$, $u_x(x,y)$, $u_y(x,y)$ for any manner in which h tends to zero.

Theorem 4. Under the conditions of Theorem 3, the solution of the boundary-value problem (1), (2) and its first derivatives depend continuously on the boundary conditions and on their first derivatives.

3°. **Theorem 5.** If f satisfies a Lipschitz condition with constant L_f in all its arguments, and φ', ψ', g' satisfy Lipschitz conditions with constants $L_{\varphi'}$, $L_{\psi'}$, $L_{g'}$, respectively, then in $\overline{G}^{**}$ the estimate

$$|\tilde{u}-u|+|\tilde{u}_x-u_x|+|\tilde{u}_y-u_y|\le \varepsilon(h)e^{L^*M(x+y)}, \tag{20}$$

where

$$\varepsilon(h) = h\{l_{u_x}(1+M_g) + l_{u_y}(1+L_{g'}) + M_{fM_{g'}} + \\ + M_{\varphi'} + L_{\varphi'} + L_f(1 + l_{u_x} + M_f)(2l_x^* + l_y^* + l_x^* l_y^*) + \\ + 2M_g^2 e^{l_x^* L_1} + 2M_{g'}^2[1 + l_{u_y} + (1+h)M_f](e^{L_1 l_x^*} - 1)\frac{L_f}{L_1}\}^*. \quad (21)$$

Moscow State University
named after M. V. Lomonosov

Received
8 VII 1957

CITED LITERATURE

[1] B. M. Budak, *Vestn. MGU*, No. 1 (1956). [2] A. D. Gorbunov, B. M. Budak, *Vestn. MGU*, No. 4 (1957). [3] B. M. Budak, A. D. Gorbunov, *DAN*, 117, No. 4 (1957).

* We recall that L_1 denotes the Lipschitz constant of the function f with respect to the arguments u_x and u_y; L_f is the Lipschitz constant of f with respect to all arguments jointly.

Note: Figure translations are in progress. See original paper for figures.